Lagrangians with reduced-order Euler–Lagrange equations

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Abstract

Any Lagrangian form of order $k$ obtained by horizontalization of a form of order $k - 1$ gives rise to Euler–Lagrange equations of order strictly less than $2k$.

But these are not the only possibilities. For example, with two independent variables, the horizontalization of a first-order 2-form gives a Lagrangian quadratic in the second-order variables; but there are also cubic second-order Lagrangians with third-order Euler–Lagrange equations.
Abstract (continued)

In this talk I shall show first that any Lagrangian of order $k$ with
Euler–Lagrange equations of order less than $2k$ must be a
polynomial in the $k$-th order variables of order not greater than the
number of different symmetric multi-indices of length $k$.

I shall then describe a geometrical construction, based on
Peter Olver’s idea of differential hyperforms, which gives rise to
Lagrangians with reduced-order Euler–Lagrange equations.

A version of this talk was given at Ostrava in June 2017. The work
has been published in $SIGMA$ 14 (2018), 089, 13 pages.
The Euler–Lagrange equations

Let $L$ be a Lagrangian in a single independent variable $x$, $n$ dependent variables $u^\alpha$, and $n$ derivative variables $u_x^\alpha$. 
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The Euler–Lagrange equations are

$$\frac{\partial L}{\partial u^\beta} - \frac{d}{dx} \frac{\partial L}{\partial u_x^\beta} = 0$$

and expanding the total derivative $d/dx$ gives

$$\frac{\partial L}{\partial u^\beta} - \frac{\partial^2 L}{\partial x \partial u_x^\beta} - u_x^\alpha \frac{\partial^2 L}{\partial u^\alpha \partial u_x^\beta} - u_x^{\alpha x} \frac{\partial^2 L}{\partial u_x^\alpha \partial u_x^\beta}$$

In general these equations are second-order, but if $L$ is linear in the variables $u_x^\alpha$ then they are first-order.
The Euler–Lagrange equations (2)

Now suppose there are $m$ independent variables $x^i$, $n$ dependent variables $u^\alpha$, and $mn$ derivative variables $u^\alpha_i$. 
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The Euler–Lagrange equations are now

$$\frac{\partial L}{\partial u^\beta} - \frac{d}{dx^j} \frac{\partial L}{\partial u^\beta_j} = 0$$

and expanding the total derivative $d/dx^j$ now gives

$$\frac{\partial L}{\partial u^\beta} - \frac{\partial^2 L}{\partial x^j \partial u^\beta_j} - u^\alpha_j \frac{\partial^2 L}{\partial u^\alpha \partial u^\beta_j} - u^\alpha_i \frac{\partial^2 L}{\partial u^\alpha_i \partial u^\beta_j}$$

In general these equations are second-order, but if $L$ is linear in the variables $u^\alpha_i$ then they are first-order. But …
The Euler–Lagrange equations (3)

\[
\frac{\partial L}{\partial u^\beta} - \frac{\partial^2 L}{\partial x^j \partial u_j^\beta} - u_j^\alpha \frac{\partial^2 L}{\partial u^\alpha \partial u_j^\beta} - u_i^\alpha \frac{\partial^2 L}{\partial u^\alpha_i \partial u_j^\beta}
\]

The equations can be first-order even when \( L \) is not linear: for example \( L = f(x, u)(u_i^\alpha u_j^\beta - u_j^\alpha u_i^\beta) \)
The Euler–Lagrange equations (3)

\[ \frac{\partial L}{\partial u^\beta} - \frac{\partial^2 L}{\partial x^j \partial u_j^\beta} - u_j^\alpha \frac{\partial^2 L}{\partial u^\alpha \partial u_j^\beta} - u_{ij}^\alpha \frac{\partial^2 L}{\partial u_i^\alpha \partial u_j^\beta} \]

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These Lagrangians come from the geometric construction of horizontalization on jet bundles:

with a fibred manifold \( \pi : E \to M \),
any differential form \( \omega \) on \( E \)
gives a horizontal differential form \( h(\omega) \) on \( J^1 \pi \)

For instance, \( h(du^\alpha \wedge du^\beta) = (u_i^\alpha u_j^\beta - u_j^\alpha u_i^\beta) dx^i \wedge dx^j \)
The Euler–Lagrange equations (4)

The same applies for higher-order Lagrangians.

If the Lagrangian $L$ has order $k$, the Euler–Lagrange equations are generically of order $2k$:

$$
\sum_{|I|=0}^k (-1)^{|I|} \frac{d^{|I|}}{dx^I} \frac{\partial L}{\partial u_I^\beta} = 0
$$

where $I \in \mathbb{N}^k$ is a symmetric multi-index:

$$
\text{if } u_I^\beta = u_{i_1 i_2 \ldots i_k}^\beta \text{ then } I(i) = |\{i_r : i_r = i\}|
$$
The Euler–Lagrange equations (4)

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\]

The geometry of the multi-index space is important:

\[
|I| = \sum_{i=1}^{m} I(i) \text{ is the length of } I;
\]

\[
\|I\|^2 = \sum_{i=1}^{m} (I(i))^2 \text{ is the square Euclidean norm of } I
\]
Reduced-order Euler–Lagrange equations

\[ \sum_{|J|=0}^k (-1)^{|J|} \frac{d^{|J|}}{dx^J} \frac{\partial L}{\partial u_\beta^J} = 0 \]

Each total derivative \( d/dx^j \) increases the order of its argument by one, so that the terms of order \( 2k \) come from

\[ \sum_{|J|=k} (-1)^k \frac{d^{|J|}}{dx^J} \frac{\partial L}{\partial u_\beta^J} \quad \text{and equal} \quad \sum_{|I|=|J|=k} (-1)^k u_\alpha^I + J \frac{\partial^2 L}{\partial u_\alpha^I \partial u_\beta^J} \]

The equations will have order less than \( 2k \) if, and only if, for each multi-index \( H \) of length \( 2k \),

\[ \sum_{I+J=H} \frac{\partial^2 L}{\partial u_\alpha^I \partial u_\beta^J} = 0 \]
The polynomial condition

Euler–Lagrange equations:
\[ \sum_{|J|=0}^{k} (-1)^{|J|} \frac{d^{|J|}}{dx^J} \frac{\partial L}{\partial u^\beta_J} = 0 \]

Condition for lower order equations: whenever \(|H| = 2k\) then
\[ \sum_{I+J=H} \frac{\partial^2 L}{\partial u^\alpha_I \partial u^\beta_J} = 0 \]

**Theorem**

A necessary condition for the Euler–Lagrange equations to have order less than \(2k\) is that \(L\) is a polynomial in the highest-order derivatives \(u^\alpha_I, |I| = k\), of order at most \(p_k\)

*where \(p_k\) is the number of distinct multi-indices of length \(k\)*
Special case: $k = 2$

$$
\sum_{|K|=2} \frac{d^2}{dx^K} \frac{\partial L}{\partial u^K} \partial L
$$

has order strictly less than 4, so that

$$
\sum_{|J|=|K|=2} u_\alpha^{J+K} \frac{\partial^2 L}{\partial u_\alpha^J \partial u^K} + \cdots
$$

has order strictly less than 4. That means

$$
\sum_{J+K=H} \frac{\partial^2 L}{\partial u_\alpha^J \partial u^K} = 0
$$

whenever $|H| = 4$ and $|J| = |K| = 2$. 
Special case: $k = 2$ (2)

$$
\sum_{J+K=H} \frac{\partial^2 L}{\partial u^\alpha_J \partial u^\beta_K} = 0
$$

whenever $|H| = 4 = 2k$ and $|J| = |K| = 2 = k$

Put $K_i = (0, \ldots, 0, 2, 0, \ldots 0)$ and $H_i = (0, \ldots, 0, 4, 0 \ldots 0)$
so that $H_i = K_i + K_i$ (‘pure’ multi-indices); then

$$
\frac{\partial^2 L}{\partial u^K_i \partial u^K_i} = 0
$$

so $L$ is at most linear in $u^K_i$
Special case: $k = 2$ (3)

$$\sum_{J+K=H} \frac{\partial^2 L}{\partial u^\alpha_J \partial u^\beta_K} = 0$$

whenever $|H| = 4$ and $|J| = |K| = 2$.

If $K_{ij} = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0)$ (‘mixed’ multi-indices)
so that $H = K_{ij} + K_{ij} = (0, \ldots, 2, 0, \ldots, 0, 2, 0, \ldots, 0)$, then

$$\frac{\partial^2 L}{\partial u^K_{ij} u^K_{Kj}} = - \frac{\partial^2 L}{\partial u^K_{ki} u^K_{Kj}} - \frac{\partial^2 L}{\partial u^K_{Kj} u^K_{ki}}$$

(we have turned ‘mixed’ into ‘pure’!)
Special case: $k = 2$ (3)

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$$\frac{\partial^2 L}{\partial u^\alpha_{K_{ij}} \partial u^\beta_{K_{ij}}} = - \frac{\partial^2 L}{\partial u^\alpha_{K_i} \partial u^\beta_{K_j}} - \frac{\partial^2 L}{\partial u^\alpha_{K_j} \partial u^\beta_{K_i}}$$

(we have turned ‘mixed’ into ‘pure’!) so that

$$\frac{\partial^4 L}{\partial u^\gamma_{K_{ih}} \partial u^\delta_{K_{ih}} \partial u^\alpha_{K_{ij}} \partial u^\beta_{K_{ij}}} = 0$$

(with $i \neq h, j$); $L$ is at most cubic in ‘overlapping’ terms $u^\alpha_{K_{ij}}$.
Special case: $k = 2$ (4)

So we know
\[
\frac{\partial^2 L}{\partial u^K_i \partial u^K_i} = 0, \quad \frac{\partial^4 L}{\partial u^{K_i h} \partial u^{K_i h} \partial u^{K_i j} \partial u^{K_i j}} = 0 \quad (i \neq h, j)
\]

If
\[
\frac{\partial^r L}{\partial u^{\alpha_1}_{J_1} \partial u^{\alpha_2}_{J_2} \cdots \partial u^{\alpha_r}_{J_r}} \neq 0
\]

then the list of multi-indices $(J_1 J_2 \cdots J_r)$ is constrained.

This implies $r \leq \frac{1}{2} m(m + 1) = p_2$,

the number of distinct multi-indices of length 2
Proof of the polynomial condition

A necessary condition for the Euler–Lagrange equations to have order less than \(2k\) is that \(L\) is a polynomial in the highest-order derivatives \(u_I^\alpha, |I| = k\), of order at most \(p_k\)

Consider

\[
\frac{\partial^{p_k+1} L}{\partial u_{J_1}^{\alpha_1} \cdots \partial u_{J_{p_k}}^{\alpha_{p_k}} \partial u_{J_{p_k+1}}^{\alpha_{p_k+1}}}
\]

so at least two of the multi-indices must be the same — say \(J_1 = J_2\)

Use the condition \(\sum_{I+J=H} \frac{\partial^2 L}{\partial u_I^\alpha \partial u_J^\beta} = 0\) to put

\[
\frac{\partial^{p_k+1} L}{\partial u_{J_1}^{\alpha_1} \partial u_{J_2}^{\alpha_2} \partial u_{J_3}^{\alpha_3} \cdots} = \sum_{K_1+K_2=J_1+J_2} \frac{\partial^{p_k+1} L}{\partial u_{K_1}^{\alpha_1} \partial u_{K_2}^{\alpha_2} \partial u_{J_3}^{\alpha_3} \cdots} - \sum_{(K_1,K_2)\neq(J_1,J_2)} \frac{\partial^{p_k+1} L}{\partial u_{K_1}^{\alpha_1} \partial u_{K_2}^{\alpha_2} \partial u_{J_3}^{\alpha_3} \cdots}
\]
Proof of the polynomial condition (2)

\[
\frac{\partial p_{k+1} L}{\partial u^{\alpha_1}_{J_1} \partial u^{\alpha_2}_{J_2} \partial u^{\alpha_3}_{J_3} \cdots} = \sum_{\substack{K_1 + K_2 = J_1 + J_2 \\ (K_1, K_2) \neq (J_1, J_2)}} - \frac{\partial p_{k+1} L}{\partial u^{\alpha_1}_{K_1} \partial u^{\alpha_2}_{K_2} \partial u^{\alpha_3}_{J_3} \cdots}
\]

But each term on the RHS also has a repeated multi-index! So we can continue ...
Proof of the polynomial condition (2)

\[
\frac{\partial p_{k+1} L}{\partial u_1^{\alpha_1} \partial u_2^{\alpha_2} \partial u_3^{\alpha_3} \cdots} = \sum_{K_1 + K_2 = J_1 + J_2} \frac{\partial p_{k+1} L}{\partial u_1^{\alpha_1} \partial u_2^{\alpha_2} \partial u_3^{\alpha_3} \cdots} - \frac{\partial p_{k+1} L}{\partial u_1^{\alpha_1} \partial u_2^{\alpha_2} \partial u_3^{\alpha_3} \cdots}
\]

But each term on the RHS also has a repeated multi-index! So we can continue . . .

But eventually, every term will have a repeated ‘pure’ multi-index \( J \) (where \( J(j) = k \) for some \( j \), and \( J(j) = 0 \) for \( i \neq j \))

and then \[ \sum_{J+J=H} \frac{\partial^2 L}{\partial u_j^\alpha \partial u_j^\beta} = 0 \] implies that

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$$\frac{\partial p_{k+1} L}{\partial u^{\alpha_1}_{J_1} \partial u^{\alpha_2}_{J_2} \partial u^{\alpha_3}_{J_3} \cdots} = \sum_{K_1+K_2=J_1+J_2} \frac{\partial p_{k+1} L}{\partial u^{\alpha_1}_{K_1} \partial u^{\alpha_2}_{K_2} \partial u^{\alpha_3}_{J_3} \cdots}$$

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$$\sum_{J+J=H} \frac{\partial^2 L}{\partial u^{\alpha}_J \partial u^{\beta}_J} = 0$$

implies that

$$\frac{\partial^2 L}{\partial u^{\alpha}_J \partial u^{\beta}_J} = 0 \quad \text{But how do we know?}$$
Proof of the polynomial condition (3)

We use the parallelogram rule for Euclidean norms!

\[ 2\|x\|^2 \leq 2\|x\|^2 + 2\|y\|^2 = \|x + y\|^2 + \|x - y\|^2 \]

with equality when \( y = 0 \),
Proof of the polynomial condition (3)

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\[
\frac{\partial^{p_k+1} L}{\partial u^{\alpha_1}_J \partial u^{\alpha_2}_J \partial u^{\alpha_3}_J \cdots} = \sum_{K_1+K_2=J+J \atop (K_1,K_2) \neq (J,J)} - \frac{\partial^{p_k+1} L}{\partial u^{\alpha_1}_{K_1} \partial u^{\alpha_2}_{K_2} \partial u^{\alpha_3}_{J_3} \cdots}
\]

we have \( \|J\|^2 + \|J\|^2 = 2\|J\|^2 < \|K_1\|^2 + \|K_2\|^2 \)

The sum of the square Euclidean norms in the terms keeps increasing ...
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\]

we have \( \|J\|^2 + \|J\|^2 = 2\|J\|^2 < \|K_1\|^2 + \|K_2\|^2 \)

The sum of the square Euclidean norms in the terms keeps increasing ... and \( \|K\|^2 = \sum (K(i))^2 \) is maximal when \( K \) is pure!

So eventually we get \( k + 1 \) pure multi-indices per term
Proof of the polynomial condition (4)

Therefore

\[
\frac{\partial^{p_k+1} L}{\partial u_1^{\alpha_1} \cdots \partial u^{\alpha_p k} \partial u^{\alpha_{p_k+1}}} = 0
\]

so that \( L \) is a polynomial in the \( u_1^{\alpha}, |J| = k \), of degree at most \( p_k \). \(\blacksquare\)
Proof of the polynomial condition (4)

Therefore

\[ \frac{\partial^{p_k+1} L}{\partial u_{J_1}^{\alpha_1} \cdots \partial u_{J_p}^{\alpha_p} \partial u_{J_p}^{\alpha_{p+1}}} = 0 \]

so that \( L \) is a polynomial in the \( u_{J}^{\alpha}, |J| = k \), of degree at most \( p_k \).

But this necessary condition is not sufficient: for instance, \( L = (u_{xy})^2 \) has Euler–Lagrange equations \( 2u_{xxyy} = 0 \)

All the Lagrangians with lower-order equations appear to be determinants

Geometrically, determinants arise as the coefficients of wedge products \( dx \wedge dy \wedge dz \wedge \cdots \)
Proof of the polynomial condition (4)

Therefore

$$\frac{\partial^p_k + 1 L}{\partial u^\alpha_1 \cdots \partial u^\alpha_{p_k} \partial u^\alpha_{p_k+1}} = 0$$

so that $L$ is a polynomial in the $u^\alpha_J$, $|J| = k$, of degree at most $p_k$.

But this necessary condition is not sufficient: for instance, $L = (u_{xy})^2$ has Euler–Lagrange equations $2u_{xxyy} = 0$.

All the Lagrangians with lower-order equations appear to be determinants.

Geometrically, determinants arise as the coefficients of wedge products $dx \wedge dy \wedge dz \wedge \cdots$

... but also as coefficients of $dx^2 \wedge dxdy \wedge dy^2 \wedge \cdots$
Differential hyperforms

Differential hyperforms were described in an unpublished paper by Peter Olver from 1982

They are covariant tensors with symmetry properties described by Young diagrams (ordinary differential forms are purely alternating, but hyperforms can have more complicated symmetries)
Differential hyperforms

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Consider hyperforms on jet manifolds $J^k \pi$ that are

- horizontal over $M$, and
- wedge products of symmetric tensors (all of the same rank)
Differential hyperforms

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Consider hyperforms on jet manifolds $J^k\pi$ that are

- horizontal over $M$, and
- wedge products of symmetric tensors (all of the same rank)

A $(p, q)$ hyperform is a section of $\bigwedge^p S^q T^* M$, pulled back to $J^k\pi$.

These are generated over $C^\infty(J^k\pi)$ by $dx^{I_1} \wedge dx^{I_2} \wedge \cdots \wedge dx^{I_p}$ where $dx^I = dx^{i_1} dx^{i_2} \cdots dx^{i_q}$ with $I = (i_1, i_2, \cdots , i_q)$.
Affine $(1, q)$ hyperforms

A $(1, q)$ hyperform $(1 \leq q \leq k)$ is a horizontal symmetric tensor

$$\theta : J^k \pi \rightarrow S^q T^* M$$

As $J^k \pi \rightarrow J^{k-1} \pi$ is an affine bundle, we say that $\theta$ is an affine $(1, q)$ hyperform if its restriction to each fibre of the bundle is an affine map: in coordinates

$$\theta = \sum_{|I|=k, |J|=q} (\theta^I_{\alpha J} u^\alpha_I + \theta_J) dx^J$$
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$$\theta = \sum_{|I|=k, |\mathcal{J}|=q} \left( \theta^I_{\alpha \mathcal{J}} u_I^\alpha + \theta_{\mathcal{J}} \right) dx^{\mathcal{J}}$$

These affine $(1, q)$ hyperforms are too general. We shall restrict attention to special affine $(1, q)$ hyperforms
Special affine \((1, q)\) hyperforms

The affine bundle \(J^k\pi \rightarrow J^{k-1}\pi\) has associated vector bundle \(V\pi \otimes S^kT^*M \rightarrow J^{k-1}\pi\)

The fibre-affine map \(\theta\) has an associated fibre-linear ‘difference map’ \(\bar{\theta} : V\pi \otimes S^kT^*M \rightarrow S^qT^*M\)

We say that \(\theta\) is a \textit{special affine} \((1, q)\) \textit{hyperform} if there is a tensor \(\tilde{\theta} \in V\pi^* \otimes S^{k-q}TM\) such that the difference map \(\bar{\theta}\) is given by contraction of elements of its domain \(V\pi \otimes S^kT^*M\) with \(\tilde{\theta}\).

In coordinates (where \(\theta^I_{\alpha}\) are the coordinates of \(\tilde{\theta}\))

\[
\theta = \sum_{|I|=k-q, |J|=q} \left(\theta^I_{\alpha} u^\alpha_{I+J} + \theta_J\right) dx^J
\]
Special affine \((1, q)\) hyperforms — example

\[ \theta = \sum_{|I|=k-q} (\theta^I_{\alpha} u^\alpha_{I+J} + \theta_J) dx^J \]

In the special case where \( q = 1 \) we have

\[ \theta = \sum_{|I|=k-1} (\theta^I_{\alpha} u^\alpha_{I+1j} + \theta_j) dx^j \]

the ordinary horizontalization of the 1-form

\[ \sum_{|I|=k-1} \theta^I_{\alpha} du^\alpha_I + \theta_j dx^j \]

There is no invariant operation of horizontalization for hyperforms when \( q \geq 2 \); but special affine \((1, q)\) hyperforms generalize the images of the horizontalization operator on ordinary 1-forms
Hyperaffine \((pq, q)\) hyperforms

A \((pq, q)\) hyperform \(\omega\) is a section of the line bundle \(\wedge^p q S^q T^* M\), pulled back to \(J^k \pi\).

It is hyperaffine if it is generated by wedge products of special affine hyperforms \(\theta = \sum_{|I|=k-q, |J|=q} (\theta^I u^\alpha_I + \theta^J) dx^J\)

If \(\omega = \omega_q dx^{I_1} \wedge dx^{I_2} \wedge \cdots \wedge dx^{I_{pq}}\) then \(\omega_q\) is a linear combination of determinants (or their minors)

\[
\begin{vmatrix}
  u^{\alpha_1}_{I_1 + J_1} & u^{\alpha_1}_{I_1 + J_2} & \cdots & u^{\alpha_1}_{I_1 + J_{pq}} \\
  u^{\alpha_2}_{I_2 + J_1} & u^{\alpha_2}_{I_2 + J_2} & \cdots & u^{\alpha_2}_{I_2 + J_{pq}} \\
  \vdots & \vdots & \ddots & \vdots \\
  u^{\alpha_{pq}}_{I_{pq} + J_1} & u^{\alpha_{pq}}_{I_{pq} + J_2} & \cdots & u^{\alpha_{pq}}_{I_{pq} + J_{pq}}
\end{vmatrix}
\]
What does this have to do with Lagrangians?

A Lagrangian $m$-form $\lambda$ defines local Lagrangian functions $L$ by

$$\lambda = L \, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^m$$
What does this have to do with Lagrangians?

A Lagrangian $m$-form $\lambda$ defines local Lagrangian functions $L$ by

$$\lambda = L \, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^m$$

Say that $\lambda$ is hyperaffine if, in any coordinate system,

$$L = \omega_1 + \omega_2 \cdots + \omega_k$$

where each $\omega_q$ is the coefficient of a hyperaffine hyperform

$$\omega = \omega_q \, dx^{J_1} \wedge dx^{J_2} \wedge \cdots \wedge dx^{J_{pq}}$$

This is independent of the coordinate system

In new coordinates $(\tilde{x}, \tilde{u})$, the volume $dx^1 \wedge dx^2 \wedge \cdots \wedge dx^m$ changes by the Jacobian determinant $J(\tilde{x}, x)$, whereas each hypervolume $dx^{J_1} \wedge dx^{J_2} \wedge \cdots \wedge dx^{J_{pq}}$ changes by a power of $J(\tilde{x}, x)$.
Euler–Lagrange equations of hyperaffine Lagrangians

Theorem

If $L$ is the Lagrangian function of a hyperaffine Lagrangian then the Euler-Lagrange equations have reduced order
Euler–Lagrange equations of hyperaffine Lagrangians

**Theorem**

If $L$ is the Lagrangian function of a hyperaffine Lagrangian then the Euler-Lagrange equations have reduced order

It is sufficient to show this for a determinant

\[
\Delta = \begin{vmatrix}
    u_{I_1}^{\alpha_1} + J_1 & u_{I_1}^{\alpha_1} + J_2 & \cdots & u_{I_1}^{\alpha_1} + J_{pq} \\
    u_{I_2}^{\alpha_2} + J_1 & u_{I_2}^{\alpha_2} + J_2 & \cdots & u_{I_2}^{\alpha_2} + J_{pq} \\
    \vdots & \vdots & \ddots & \vdots \\
    u_{I_{pq}}^{\alpha_{pq}} + J_1 & u_{I_{pq}}^{\alpha_{pq}} + J_2 & \cdots & u_{I_{pq}}^{\alpha_{pq}} + J_{pq}
\end{vmatrix}
\]

so write $\Delta$ as

\[
\Delta = \sum_{\sigma \in \mathcal{S}_n} \varepsilon_{\sigma} u_{I_1}^{\alpha_1} + J_{\sigma(1)} u_{I_2}^{\alpha_2} + J_{\sigma(2)} \cdots u_{I_h} + J_{\sigma(h)}
\]
Euler–Lagrange equations of hyperaffine Lagrangians (2)

$$\Delta = \sum_{\sigma \in \mathcal{S}_h} \varepsilon_\sigma u^{\alpha_1}_I + J_{\sigma(1)} u^{\alpha_2}_I + J_{\sigma(2)} \cdots u^{\alpha_h}_I + J_{\sigma(h)}$$

Substituting in the Euler–Lagrange equations gives

$$\sum_{|K|=k} \frac{d|K|}{dx^K} \frac{\partial L}{\partial u^K_\beta} = \sum_{1 \leq r, s \leq h} \sum_{\sigma \in \mathcal{S}_h} \delta^\alpha_r \varepsilon_\sigma \Phi_{r,s,\sigma} u^{\alpha_s}_I \big|_{r \neq s} + J_{\sigma(r)} + J_{\sigma(s)}$$

where the coefficients $\Phi_{r,s,\sigma}$ are

$$\Phi_{r,s,\sigma} = u^{\alpha_1}_I + J_{\sigma(1)} u^{\alpha_2}_I + J_{\sigma(2)} \cdots \hat{r} \cdots \hat{s} \cdots u^{\alpha_h}_I + J_{\sigma(h)}$$
Euler–Lagrange equations of hyperaffine Lagrangians (2)

\[
\Delta = \sum_{\sigma \in \mathcal{G}_h} \varepsilon_{\sigma} u_{I_1}^{\alpha_1} + J_{\sigma(1)} u_{I_2}^{\alpha_2} + J_{\sigma(2)} \cdots u_{I_h}^{\alpha_h}
\]

Substituting in the Euler–Lagrange equations gives

\[
\sum_{|K|=k} \frac{d|K|}{dx^K} \frac{\partial L}{\partial u^K_\beta} = \sum_{1 \leq r, s \leq h} \sum_{\sigma \in \mathcal{G}_h} \delta_{\beta}^{\alpha_r} \varepsilon_{\sigma} \Phi_{rs\sigma} u_{I_r}^{\alpha_s} + I_s + J_{\sigma(r)} + J_{\sigma(s)}
\]

where the coefficients \( \Phi_{rs\sigma} \) are

\[
\Phi_{rs\sigma} = u_{I_1}^{\alpha_1} + J_{\sigma(1)} u_{I_2}^{\alpha_2} + J_{\sigma(2)} \cdots \hat{r} \cdots \hat{s} \cdots u_{I_h}^{\alpha_h}
\]

Fix \( r \neq s \). Given \( \sigma \in \mathcal{G}_h \) put \( \tilde{\sigma} = \sigma \circ (r, s) \neq \sigma \).
\( \Phi_{rs\sigma} = \Phi_{rs\tilde{\sigma}} \) and \( \varepsilon_{\sigma} = -\varepsilon_{\tilde{\sigma}} \) so all the terms cancel.
Established so far:

- If a Lagrangian function of order $k$ has reduced-order Euler–Lagrange equations then it is a polynomial of order at most $p_k$ in the variables $u^\alpha_H (|H| = k)$;
- Every hyperaffine Lagrangian has reduced-order Euler–Lagrange equations (and is a polynomial with a particular determinant structure)

I conjecture that every Lagrangian with reduced-order Euler–Lagrange equations has this particular determinant structure, and so is hyperaffine.
Determinants (2)

A general polynomial Lagrangian function of order $k$ and degree $p_k$ is

$$L = \sum_{r=0}^{p_k} A_{\alpha_1 \alpha_2 \cdots \alpha_r}^{H_1 H_2 \cdots H_r} u^{\alpha_1}_{H_1} u^{\alpha_2}_{H_2} \cdots u^{\alpha_r}_{H_r}$$

with implicit sums over the indices and multi-indices, and with $|H| = k$.

Can this be written as a linear combination of determinants

$$\begin{vmatrix} u^{\alpha_1}_{I_1 + J_1} & u^{\alpha_1}_{I_1 + J_2} & \cdots & u^{\alpha_1}_{I_1 + J_r} \\ u^{\alpha_2}_{I_2 + J_1} & u^{\alpha_2}_{I_2 + J_2} & \cdots & u^{\alpha_2}_{I_2 + J_r} \\ \vdots & \vdots & \ddots & \vdots \\ u^{\alpha_r}_{I_r + J_1} & u^{\alpha_r}_{I_r + J_2} & \cdots & u^{\alpha_r}_{I_r + J_r} \end{vmatrix}$$

$|J| = q$, $|I| = k - q$

$1 \leq q \leq k$, $0 \leq r \leq p_q$?
Determinants (3)

Consider homogeneous polynomials \(A_{\alpha_1 \alpha_2 \cdots \alpha_r}^{H_1 H_2 \cdots H_r} u_{H_1}^{\alpha_1} u_{H_2}^{\alpha_2} \cdots u_{H_r}^{\alpha_r}\)

In the case \(r = 2\) there is a constructive proof
Determinants (3)

Consider homogeneous polynomials
\[ A^{H_1 H_2 \ldots H_r} u^{\alpha_1}_{H_1} u^{\alpha_2}_{H_2} \cdots u^{\alpha_r}_{H_r} \]

In the case \( r = 2 \) there is a constructive proof

Partition the quadratic terms by \( H_1 + H_2 = H \) and put

\[ \psi_H = \sum_{H_1 + H_2 = H} A^{H_1 H_2} u^{\alpha_1}_{H_1} u^{\alpha_2}_{H_2} \]

Choose a term \( A^{K_1 K_2} u^{\alpha_1}_{K_1} u^{\alpha_2}_{K_2} \) arbitrarily, so from E–L we have

\[ A^{K_1 K_2} = \sum_{H_1 + H_2 = H, (H_1, H_2) \neq (K_1, K_2)} - A^{H_1 H_2} \]

and so

\[ \psi_H = \sum_{H_1 + H_2 = H} A^{H_1 H_2} (u^{\alpha_1}_{H_1} u^{\alpha_2}_{H_2} \ - \ u^{\alpha_1}_{K_1} u^{\alpha_2}_{K_2}) \]
Determinants (4)

For cubic and higher terms, there is no obvious algorithm to give an explicit construction

(although ad-hoc methods work for all examples investigated)
Determinants (4)

For cubic and higher terms, there is no obvious algorithm to give an explicit construction

(although ad-hoc methods work for all examples investigated)

A possible approach would use an abstract dimension argument:

The number of variables $u_I^\alpha$, $|I| = k$, is known, and so the dimension of the space of homogeneous polynomials of degree $r$ is also known

The number of E–L constraints for quadratic polynomials is known, so the number of constraints for degree $r$ polynomials can in principle be calculated

The theorem will be proved if there are enough independent $r \times r$ determinants of the correct type