An IDEAL characterization of cosmological and black hole spacetimes
(cf. arXiv:1704.05542,1807.09699, pub. in CQG)

Igor Khavkine

Institute of Mathematics
Czech Academy of Sciences, Prague

Ostrava Seminar on Mathematical Physics
University of Ostrava
20 Mar 2019
Motivation

- The **fundamental symmetries** in General Relativity (GR) are **diffeomorphisms**.
- Two (Lorentzian) spacetime geometries \((M, g)\) and \((M, g')\) may appear to be very different but still be related by a diffeomorphism. The geometries are **isometric**.
- A lot of effort can go into deciding whether two geometries belong to the same (local) isometry class.

**Definition (locally isometric)**

\((M, g)\) is **locally isometric** to \((N, h)\) if \(\forall x \in M \exists y \in N\) such that a neighborhood of \(x\) is isometric to a neighborhood of \(y\). All such \((M, g)\) constitute the local isometry class of \((N, h)\).
Q: Given a model geometry \((N, h)\), is it possible to verify when \((M, g)\) belongs to its local isometry class by checking a list of equations

\[
T_a[g] = 0 \quad (a = 1, 2, \ldots, A),
\]

where each \(T_a[g]\) is a tensor covariantly constructed from \(g\) and its derivatives?

If Yes, we call this an IDEAL (Intrinsic, Deductive, Explicit, ALgorithmic) characterization of the local isometry class of \((N, h)\). Sometimes, also called Rainich-like.

Generalizes to \((M, g, \Phi)\), including matter (tensor) fields, if we use covariant tensor equations of the form \(T_a[g, \Phi] = 0\).

An alternative to the Cartan-Karlhede moving-frame-based characterization.

Also, the linearizations \(T_a[g + \varepsilon p] = T_a[g] + \varepsilon \dot{T}_a[g; p] + O(\varepsilon^2)\) constitute a complete list of local gauge invariant observables \(T_a[h; -]\) for linearized GR on \((N, h)\).
Examples:

- **Very few examples** of IDEAL characterizations are actually known. To my knowledge, they are either classic or due to the work of Ferrando & Sáez (València).

  - **Examples:**
    - **Constant curvature** (1800s): \( R = R[g] \) — Riemann tensor,
      
      \[
      R_{ijkh} = k(g_{ik}g_{jh} - g_{jk}g_{ih})
      \]
    - **Schwarzschild** of mass M in 4D (1998): \( W = W[g] \) — Weyl tensor,
      
      \[
      R_{ij} = 0, \quad S_{ijlm}S_{lk}^{\,lm} + 3\rho S_{ijkh} = 0,
      \]
      \[
      P_{ab} = 0, \quad \rho/\alpha^{3/2} - M = 0,
      \]
      
      where
      \[
      \rho = -\left(\frac{1}{12} \text{tr} \, W^3\right)^{1/3}, \quad S_{ijkh} = W_{ijkh} - \frac{1}{6}(g_{ik}g_{jh} - g_{jk}g_{ih}),
      \]
      \[
      \alpha = \frac{1}{9} (\nabla \ln \rho)^2 - 2\rho, \quad P_{ij} = (*W)_{ij}^{\,kh} \nabla_k \rho \nabla_h \rho.
      \]
    - **NEW:** FLRW, inflationary, Schwarzschild-Tangherlini

Fix a class of reference geometries \((M, g_0(\beta))\), with parameters \(\beta\).

Suppose there already exists a characterization of this class by the existence of tensor fields \(\sigma\) satisfying equations

\[
S_a[g, \sigma] = 0,
\]
covariantly constructed from \(\sigma, g_{ij}, R_{ijkl}\) and their covariant derivatives.

Exploiting the geometry of \((M, g_0(\lambda))\), we try to find formulas for \(\sigma = \Sigma[g_0]\) covariantly constructed from \(g_{ij}, R_{ijkl}\) and their covariant derivatives. If successful, we get an IDEAL characterization of this class by

\[
T_a[g] := S[g, \Sigma[g]] = 0.
\]

If necessary, find further covariant expressions for the parameters \(\beta = B[g_0]\), adding equations \(B[g] - \beta = 0\) to the above list, until we can IDEALly characterize individual isometry classes.
Let $\dim M = m + 1$.

$\mathbf{(M, g)}$ is (locally) **FLRW** when around every point of $M$ there exist local coordinates $(t, x_1, \ldots, x_m)$, such that

(a) $g_{ij}(t, x_1, \ldots, x_m) = -(dt)^2 + f^2(t)h_{ij}(x_1, \ldots, x_m)$ (warped product),

(b) $h_{ij}$ is of constant curvature (homogeneous and isotropic), e.g.

$$h_{ij} = \frac{1}{(1 - \alpha r^2)}(dr)^2 + r^2 d\Omega^2_{ij}, \quad \text{with} \quad \mathcal{R}[h] = m(m-1)\alpha.$$

$\mathbf{(M, g, \phi)}$ is (locally) **inflationary** when it is locally FLRW and the local coordinates $(t, x_1, \ldots, x_m)$ can be chosen so that the scalar $\phi = \phi(t)$, while also satisfying the **Einstein-Klein-Gordon** equations

$$R_{ij} - \frac{1}{2} \mathcal{R} g_{ij} = \kappa \left( \nabla_i \phi \nabla_j \phi - \frac{1}{2} g_{ij} [ (\nabla \phi)^2 + V(\phi) ] \right)$$

with some potential $V(\phi)$ and $\kappa \sim$ Newton’s constant.
Warped $m+1$ Products

Without constant spatial curvature, an FLRW geometry is called a Generalized Robertson Walker (GRW) geometry.

**Theorem (Sánchez, 1998)**

$(M, g)$ is locally GRW iff $\exists U$ — unit timelike vector field satisfying

$$\mathcal{P}_{jk} := U_j \nabla^k \frac{\nabla^i U_i}{m} = 0, \quad \mathcal{D}_{ij} := \nabla_i U_j - \frac{\nabla^k U^k}{m} (g_{ij} + U_i U_j) = 0.$$ 

**Theorem (Chen, 2014)**

$(M, g)$ is locally GRW iff $\exists v, \mu$ — timelike vector field and scalar satisfying $\nabla_i v_j = \mu g_{ij}$.

In coordinates, $U^i = (\partial_t)^i$ and $v^i = f(t) U^i$, meaning $U = v / \sqrt{-v^2}$.

In GRW pre-history, Sánchez’s conditions were known and stated as follows: $U$ is unit, geodesic, shear-free, twist-free and has spatially-constant expansion (Ehlers, 1961), (Easley, 1991).
Constant Spatial Curvature

▶ Convenient to define the Kulkarni-Nomizu product:

\[(A \odot B)_{ijkh} = A_{ik}B_{jh} - A_{jk}B_{ih} - A_{ih}B_{jk} + A_{jh}B_{ik}.\]

▶ Given Sánchez’s \(U^i\), define \(\xi := \frac{\nabla^i U_i}{m}\), \(\eta := -U^i \nabla_i \xi\). Eventually, \(U\) is one of normalized \(\nabla R\), \(\nabla (B := R_{ij} R^{ij})\) or \(\nabla \phi\).

▶ Spatial Zero Curvature Deviation (ZCD) tensor:

\[Z_{ijkh} := R_{ijkh} - \left(g \odot \left[\frac{\xi^2}{2} g - \eta U U\right]\right)_{ijkh}, \quad \zeta := \frac{Z_{iikk}}{m(m-1)}.\]

▶ Spatial Constant Curvature Deviation (CCD) tensor:

\[\mathcal{C}_{ijkh} := R_{ijkh} - \left(g \odot \left[\frac{(\xi^2 + \zeta)}{2} g - (\eta - \zeta) U U\right]\right)_{ijkh}.\]

▶ \(Z_{ijkh} = 0 \implies \text{flat FLRW}\).

\(\mathcal{C}_{ijkh} = 0, \ U_{[i} \nabla_j] \zeta = 0 \implies \text{generic FLRW (any curvature)}.\)
FLRW Scale Factor

➤ Scale factor derivatives: \( \xi = \frac{f'}{f}, \quad \eta = \frac{f''}{f} - \frac{f'^2}{f^2}, \quad \zeta = \frac{\alpha}{f^2} \).

➤ **Perfect fluid** interpretation: \( \rho \) — pressure, \( \rho \) — energy density,

\[
R_{ij} - \frac{1}{2} R g_{ij} + \Lambda g_{ij} = \kappa (\rho + p) U_i U_j + \kappa p g_{ij},
\]

reduces to the **Friedmann** and **acceleration** equations

\[
\xi^2 + \zeta = \frac{2}{m(m-1)} \kappa \rho, \quad \eta - \zeta = -\frac{1}{m-1} \kappa (\rho + p).
\]

➤ Flat FLRW with \( \zeta = 0, (f'/f)' \neq 0 \): can find \( P(u) \) such that

\[
\eta + \frac{m}{2} \xi^2 = -\kappa P(\xi^2).
\]

➤ Generic FLRW with \( f'/f \neq 0 \): can find \( E(u) \) such that

\[
\xi^2 + \zeta = \kappa E(\zeta).
\]

➤ ODEs in \( f \) (with parameter \( \alpha \)) **fix scale factor** up to \( (f(t), \alpha) \mapsto (Af(t + t_0), A\alpha), \) exhausting isometric \( (f(t), \alpha) \) pairs.
Flowchart: FLRW Characterization

\[ (M, g), \dim M = m + 1 \]

- \((\nabla R)^2 < 0\) → \(U := U_R\)
- \((\nabla B)^2 < 0\) → \(R - \frac{\xi}{2}(g \circ g) = 0\)
- \(\nabla_i R_{jk} = 0\) → \(CC^\Omega_{K}\)
- \(W_{jkh} = 0\)
- \(R_i (R_{jk} - (m - 1)Kg_{jk}) = 0\)
- \(\nabla_i R_{jk} = 0\)
- \(\nabla_i \nabla_j U = \xi g_{ij}\)
- \(\zeta = 0\)
- \(\nabla_i \nabla_j U = \xi g_{ij}\)
- \(\zeta = 0\)
- \(\eta + \frac{m}{2} \xi^2 = -\kappa P(\xi^2)\)
- \(\kappa P(u) = \frac{(m+1)}{2} (u - K)\)
- \(\zeta^2 + \zeta = \kappa E(\zeta)\)
- \(\kappa E(u) = K + \Omega |u|^{\frac{m+1}{2}}\)

- \(FLRW_{m,0}^{\Omega,0}\)
- \(CSC_{K,0}^{m,0}\)
- \(FLRW_{E,0}^{m,0}\)
- \(CSC_{K,\Omega,0}^{m,0}\)

Not FLRW

Igor Khavkine  (CAS, Prague)  IDEAL characterization  20/03/2019  9/15
Inflationary Scale Factor

- Scale factor derivatives: \( \xi = \frac{f'}{f} \), \( \eta = \frac{f''}{f^2} - \frac{f'^2}{f^2} \), \( \zeta = \frac{\alpha}{f^2} \).

- Einstein-Klein-Gordon equations reduce to

\[
\xi^2 + \zeta = \kappa \frac{\phi'^2 + V(\phi)}{m(m-1)}, \quad \eta - \zeta = -\kappa \frac{\phi'^2}{(m-1)}.
\]

- Flat inflationary with \( \zeta = 0, \phi' \neq 0 \): can find \( \Xi(u) \) such that

\[
\left( \text{"Hamilton-Jacobi" eq.} \right) \quad (\partial_u \Xi(u))^2 - \kappa \frac{m\Xi^2(u)}{(m-1)} + \kappa^2 \frac{V(u)}{(m-1)^2} = 0,
\]

\[
\phi' = -\frac{(m-1)}{\kappa} \partial_\phi \Xi(\phi), \quad \xi = \Xi(\phi).
\]

- Generic inflationary with \( \phi' \neq 0 \): can find \( \Xi(u), \Pi(u) \) such that

\[
\text{(new?)} \quad \Pi \left( \partial_u \Xi + \kappa \frac{\Pi}{m(m-1)} \right) - \left( \kappa \frac{\Pi^2 + V}{m(m-1)} - \Xi^2 \right) = 0,
\]

\[
\partial_u \left( \kappa \frac{\Pi^2 + V}{m(m-1)} - \Xi^2 \right) + 2 \Xi \frac{\Xi}{\Pi} \left( \kappa \frac{\Pi^2 + V}{m(m-1)} - \Xi^2 \right) = 0,
\]

\[
\phi' = \Pi(\phi), \quad \xi = \Xi(\phi).
\]

- ODEs in \((f, \phi)\) fix scale factor and inflaton up to \((f(t), \phi(t)) \mapsto (Af(t + t_0), \phi(t + t_0))\), exhausting isometric \((f(t), \phi(t))\) pairs.
Flowchart: Inflationary Characterization

\[(M, g, \phi), \dim M = m + 1 > 2\]

- \((\nabla \phi)^2 < 0\)
  - no\n  - yes
    - \(U := U_J\)

- \(\nabla_i U = 0\)
  - no
  - yes
    - \(K := \frac{2\kappa m_0}{m(m-1)}\)

- \(R - \frac{\Lambda}{m(m-1)} (g \odot g) = 0\)
  - \(\phi = \Phi\)
    - yes
    - no
      - not inflationary

- \(\nabla_i U - \frac{\nabla^2 U}{m_0} U - \xi g_i = 0\)
  - \(\xi \neq 0\)
    - yes
    - no
      - not inflationary

- \(\xi = 0\)
  - yes
  - no
    - not inflationary

- \(V = \frac{2\Lambda}{m}\)

Igor Khavkine (CAS, Prague)

IDEAL characterization

20/03/2019 11/15
Schwarzschild-Tangherlini Spacetimes

(cf. arXiv:1807.09699) Let \( n = m + 2 \).

\((M, \bar{g})\) is locally \( g_{ST} \) (generalize Schwarzschild-Tangherlini) when a neighborhood of every point of \( M \) is isometric to a portion of the maximal analytic extension of \( \bar{g}_{ij} \), where

\( (a) \quad \bar{g}_{ij} = g_{ij} + r^2 \Omega_{ij} \) (warped product),

\( (b) \quad \Omega_{ij} \) is an \( m \)-dimensional constant curvature metric, with sectional curvature \( \alpha \),

\( (c) \quad g_{ij} = -f(r) dt_i dt_j + \frac{1}{f(r)} dr_i dr_j \), with \( f(r) = \alpha - \frac{2\mu}{r^{n-3}} - \frac{2\Lambda}{(n-1)(n-2)} r^2 \).

These are higher dimensional versions of the spherically symmetric Schwarzschild black hole of mass \( \mu \), cosmological constant \( \Lambda \).
Warped $m + 2$ products

Without any constant curvature condition, such warped products can be characterized as follows.

**Theorem (García-Parrado, 2006; Ferrando-Sáez, 2010)**

$(M, \bar{g})$ is locally $\bar{g}_{ij} = g_{ij} + r^2 \Omega_{ij}$ iff $\exists \ell_i, \bar{\Omega}_{ij}$ satisfying

\[
\nabla_i [\ell_j] = 0, \quad \ell^i \bar{\Omega}_{ij} = 0,
\]

\[
\bar{\Omega}_i^j \bar{\Omega}_{jk} = \bar{\Omega}_{ik}, \quad \bar{\Omega}_i^i = m, \quad \bar{\nabla}_i \bar{\Omega}_{jk} = -2 \bar{\Omega}_{(i} \ell_{j)k}.
\]

Then $\ell_i = \bar{\nabla}_i \log |r|$, $\Omega_{ij} = r^{-2} \bar{\Omega}_{ij}$ and $g_{ij} = \bar{g}_{ij} - \bar{\Omega}_{ij}$.

In addition, it is well known that a **spherically symmetric vacuum** is locally gST. More generally, we have

**Theorem (extended Birkhoff’s theorem)**

If $(M, \bar{g})$ is a warped $2 + m$ product and $\bar{R}_{ij} - \frac{1}{2} \bar{\nabla} \bar{g}_{ij} + \Lambda \bar{g}_{ij} = 0$, then it is locally gST.

A detailed proof is given in Prop. 6 of my arXiv:1807.09699.
gST Characterization

It remains only to express the tensors $\ell_i$, $\bar{\Omega}_{ij}$ and the parameters $\Lambda$, $\mu$, $\alpha$ in terms of the curvature of a gST geometry $(M, \bar{g})$.

**Theorem (IK)**

Let $(M, \bar{g})$ be a given gST geometry. Then, letting

$$\bar{T}_{ijkl} := \bar{R}_{ijkl} - \frac{2\Lambda}{(n-1)(n-2)} (\bar{g}_{[k} \bar{g}_{l]j}), \quad \rho := \left[ \frac{(\bar{T} \cdot \bar{T} \cdot \bar{T})_{ij}}{8(n-1)(n-2)(n-3)[(n-2)(n-3)(n-4) + 2]} \right]^{\frac{1}{3}},$$

$$\ell_i := -\frac{1}{(n-1)} \bar{\nabla}_i \rho, \quad A := \ell_i \ell^i + 2\rho + \frac{2\Lambda}{(n-1)(n-2)},$$

$$\bar{\Omega}_{ij} := \frac{2(n-2)^2}{(n-1)(n-4)} (\bar{T} \cdot \bar{T})_{ik}^k \frac{(n-2)(n-3)}{(n-1)(n-4)} \bar{g}_{ij}, \quad g_{ij} := \bar{g}_{ij} - \bar{\Omega}_{ij},$$

$$Z_{ijkl} := \bar{T}_{ijkl} - \rho \left[ \frac{(n-2)(n-3)}{2} (g \circ g)_{ijkl} + (\bar{\Omega} \circ \bar{\Omega})_{ijkl} - (n-3)(g \circ \bar{\Omega})_{ijkl} \right],$$

we have $Z_{ijkl} = 0$, $\ell_i$, $\bar{\Omega}_{ij}$ satisfy the warped product conditions, while

$$\Lambda = \frac{2n\bar{R}}{(n-2)}, \quad \text{and} \quad A^{n-1} \rho^{-2} = \alpha^{n-1} \mu^{-2}.$$ 

Here, $\alpha^{n-1} \mu$ is an invariant combination of $\alpha$ and $\mu$. 
An IDEAL characterization of the (local) isometry class of a physically interesting spacetime is a natural problem of geometric interest.

FLRW, inflationary and gST spacetimes are now on the (currently short) list of IDEAL-ly characterized geometries.

Does an IDEAL characterization exist whenever a Cartan-Karlhede characterization exists?

Next steps:
- Bianchi (homogeneous) cosmologies?
- Kasner singular solutions?
- Higher dimensional rotating Myers-Perry black holes?
Discussion

- An IDEAL characterization of the (local) isometry class of a physically interesting spacetime is a natural problem of geometric interest.
- FLRW, inflationary and gST spacetimes are now on the (currently short) list of IDEAL-ly characterized geometries.
- Does an IDEAL characterization exist whenever a Cartan-Karlhede characterization exists?
- Next steps:
  - Bianchi (homogeneous) cosmologies?
  - Kasner singular solutions?
  - Higher dimensional rotating Myers-Perry black holes?

Thank you for your attention!