Ellis semigroups associated with Delone sets

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2015
Joint with Johannes Kellendonk
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Object: Classify quasicrystals.
Delone sets and their hulls

- A subset $\Lambda \subset \mathbb{R}^n$ is a **Delone set** if it is both relatively dense and uniformly discrete.
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- Any finite subset of a Delone set $\Lambda$ is called a **patch** of $\Lambda$. 

**Hull**

The **hull**, $\Omega_\Lambda$, of a Delone set $\Lambda$ is the collection of all Delone sets $\Lambda'$ with the property that each patch of $\Lambda'$ is a translate of some patch of $\Lambda$:

$$\Omega_\Lambda := \{ \Lambda' : \forall x, \forall r, \exists y, \exists t : B_r(x) \cap \Lambda' = (B_r(y) \cap \Lambda) - t \}.$$ 

In the **local topology** on $\Omega_\Lambda$, $\Lambda'$ and $\Lambda''$ are close if there is a large $r$ so that the patches $B_r(0) \cap \Lambda'$ and $B_r(0) \cap \Lambda''$ are Hausdorff close. This is a metric topology with metric denoted by $d$. 

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Ellis and Delone
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Regularity assumptions

- $\Lambda$ has **finite local complexity (FLC)** provided, for each $r$, there are only finitely many distinct patches $B_r(x) \cap \Lambda$, $x \in \mathbb{R}^n$, up to translation.
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- λ is **repetitive** if for each \(x, r\) there is \(R\) so that for each \(y\) there is \(t\) with \((B_r(x) \cap \Lambda) - t \subset B_R(y) \cap \Lambda\).

A Delone set \(\Lambda\) is a **Meyer set** if \(\Lambda - \Lambda\) is uniformly discrete.

Meyer sets automatically have FLC.

**STANDING ASSUMPTION (SA):** \(\Omega = \Omega_{\Lambda}\) is the hull of a repetitive Meyer set \(\Lambda\).

(Note that, by repetitivity, \(\Omega_{\Lambda} = \Omega_{\Lambda'}\) for all \(\Lambda' \in \Omega\).)

**Theorem:** Under SA, \((\Omega, \mathbb{R}^n)\) is a compact, minimal dynamical system.
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Topology of Ω_Λ

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Algebra of $\mathcal{E}((\Omega_\Lambda, \mathbb{R}^n))$
ELLIS SEMIGROUP:
Given a compact, metric, minimal dynamical system \((X, G)\) and \(g \in G\), let \(t_g : X \to X\) be the homeomorphism \(t_g(x) := g \cdot x\). Then \(g \mapsto t_g\) embeds \(G\) in \(X^X\). Give \(X^X\) the product topology. Then
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\mathcal{E}((X, G)) := \{t_g : g \in G\} \subset X^X,
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Typically, most elements of \(\mathcal{E}\) are neither one-to-one nor onto, are not continuous, and \(\mathcal{E}\) is nonabelian.
But \(\mathcal{E}\) is functorial and minimal left ideals exist, as do minimal idempotents.
PROXIMALITY:
Elements $\Lambda, \Lambda' \in \Omega$ are **proximal**, denoted $\Lambda \sim_p \Lambda'$, if for all $r$ there is $x$ so that

$$B_r(x) \cap \Lambda = B_r(x) \cap \Lambda'.$$

(Under SA, this is the same as saying that

$$\inf_{t \in \mathbb{R}^n} d(\Lambda - t, \Lambda' - t) = 0.$$
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$$B_r(0) \cap \Lambda = B_r(0) \cap S,$$

$$B_r(0) \cap \Lambda' = B_r(0) \cap S'$$ and

$$B_r(x) \cap S = B_r(x) \cap S'.$
MAXIMAL EQUICONTINUOUS FACTOR:

\( \sim_{rp} \) is a closed equivalence relation. Let

\[ \Omega_{max} := \Omega / \sim_{rp} \]

with quotient map

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**THEOREM** (B., Kellendonk): Under SA \((\Omega_{\text{max}}, \mathbb{R}^n)\) is a Kronecker action on a torus or solenoid. There are \(m < M\) so that:

(i) \(m \leq \#(\pi^{-1}(z)) \leq M\) for all \(z \in \Omega_{\text{max}}\);

(ii) \(\pi\) is a.e. \(m\)-to-one; and

(iii) for each \(z \in \Omega_{\text{max}}\) there are \(\Lambda_1, \ldots, \Lambda_m \in \pi^{-1}(z)\) with \(\Lambda_i \cap \Lambda_j = \emptyset\) for \(i \neq j\).
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\(\text{cr}(\Omega) := m\) is called the *coincidence rank* of \(\Omega\).
EXAMPLE:
Fibonacci.
From now on, let’s assume, in addition to SA, that $\Lambda$ is nonperiodic.

Let

$$\mathcal{E} := \mathcal{E}((\Omega, \mathbb{R}^n)).$$

Recall that $t \mapsto (\Lambda \mapsto \Lambda - t)$ embeds $\mathbb{R}^n$ into $\mathcal{E}$ (call the image $\mathbb{R}^n$ with $\overline{\mathbb{R}^n} = \mathcal{E}$). Let

$$\mathcal{E}^\infty := \mathcal{E} \setminus \mathbb{R}^n$$

and let

$$\mathcal{E}_{max} := \mathcal{E}((\Omega_{max}, \mathbb{R}^n)).$$

Then $\mathcal{E}_{max}$ is naturally isomorphic with $\Omega_{max}$ and $\pi$ induces $\pi_* : \mathcal{E}^\infty \to \mathcal{E}_{max} = \Omega_{max}$, leading to the short exact sequence:

$$1 \to K \to \mathcal{E}^\infty \to \Omega_{max} \to 1,$$

$$K := \ker(\pi_*).$$
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- $\#(f(\pi^{-1}(z))) \geq m$ for all $f \in \mathcal{E}$ and all $z \in \Omega_{\text{max}}$. 
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that if \( m = \text{cr}(\Omega) \) then

- \( \#(f(\pi^{-1}(z))) \geq m \) for all \( f \in \mathcal{E} \) and all \( z \in \Omega_{\text{max}} \).
  It follows from minimality that

- If \( \mathcal{M} \) is a minimal left ideal in \( \mathcal{E} \) then, for each \( z \in \Omega_{\text{max}} \) and \( f \in \mathcal{M} \),
  \[ \#(f(\pi^{-1}(z))) = m. \]
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It follows from minimality that

\[ \text{If } \mathcal{M} \text{ is a minimal left ideal in } \mathcal{E} \text{ then, for each } z \in \Omega_{\text{max}} \text{ and } f \in \mathcal{M}, \]
\[ \#(f(\pi^{-1}(z))) = m. \]

Furthermore, given \( f \in \mathcal{E} \) with \( \#(f(\pi^{-1}(z))) = m \) for all
\( z \in \Omega_{\text{max}}, \)

\[ \mathcal{M} = \{ g \in \mathcal{E}^{\infty} : g(\Lambda) = g(\Lambda') \iff f(\Lambda) = f(\Lambda') \} \]

is a minimal left ideal.
IDEMPOTENTS:

- Given $\Lambda, \Lambda' \in \Omega$, there is $f \in \mathcal{E}$ with $f(\Lambda) = f(\Lambda')$ if and only if $\Lambda \sim_p \Lambda'$.
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- Given $\Lambda, \Lambda' \in \Omega$, there is $f \in \mathcal{E}$ with $f(\Lambda) = f(\Lambda')$ if and only if $\Lambda \sim_p \Lambda'$.

- Moreover, if $\Lambda \sim_p \Lambda'$, and $\Lambda \neq \Lambda'$ there is a *minimal idempotent* $p \in \mathcal{E}^\infty$ ($pp = p$ and $p$ in some minimal $\mathcal{M}$) with $p(\Lambda) = p(\Lambda')$. 
**THEOREM:** If $M$ is a minimal left ideal in $\mathcal{E}$ and $\mathcal{J} \subset M$ is the set of idempotents in $M$, then:

(i) $pM$ is a group for each $p \in \mathcal{J}$ and

(ii) $M = \bigcup_{p \in \mathcal{J}} pM$ is a disjoint union.
THEOREM: If $\mathcal{M}$ is a minimal left ideal in $\mathcal{E}$ and $\mathcal{J} \subset \mathcal{M}$ is the set of idempotents in $\mathcal{M}$, then:
(i) $p\mathcal{M}$ is a group for each $p \in \mathcal{J}$ and
(ii) $\mathcal{M} = \bigcup_{p \in \mathcal{J}} p\mathcal{M}$ is a disjoint union.

Now, restricting the previous SES, we have, for each minimal idempotent $p$, an SES of groups:

$$1 \to K_p \to p\mathcal{M} \to \Omega_{max} \to 1,$$

In which the the kernel $K_p$ is trivial if and only if $cr = 1$. 
EXAMPLE:
Let $\Lambda$ be the previously described Fibonacci point set. Then $\Omega_{\text{max}} = \mathbb{T}^2$, $\mathcal{E}$ has a unique minimal left ideal, which equals $\mathcal{E}^\infty$, and two minimal idempotents $p, q$. The kernels $K_p = \{p\}$ and $K_q = \{q\}$ of the preceding SES are trivial, so that

$$p\mathcal{E}^\infty \simeq \mathbb{T}^2 \simeq q\mathcal{E}^\infty$$

giving the description

$$\mathcal{E}^\infty \simeq \{p, q\} \times \mathbb{T}^2$$

where multiplication in the product is given by

$$(x, z) \cdot (y, w) := (x, z + w).$$
The point set $\Lambda$ is a *perfect quasicrystal* if $cr(\Omega_\Lambda) = 1$. (This is the case if and only if $\Lambda$ has pure point diffraction spectrum.)
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**THEOREM:** $\mathcal{E}((\Omega_\Lambda, \mathbb{R}^n))$ has a unique minimal left ideal $\mathcal{M}$ if and only if $cr(\Omega) = 1$ ($\Lambda$ is a ‘perfect quasicrystal’). In this case,

$$\mathcal{M} \simeq \mathcal{J} \times \Omega_{\text{max}}$$

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The point set \( \Lambda \) is a \textit{perfect quasicrystal} if \( cr(\Omega_\Lambda) = 1 \). (This is the case if and only if \( \Lambda \) has pure point diffraction spectrum.)

**THEOREM:** \( \mathcal{E}(\Omega_\Lambda, \mathbb{R}^n) \) has a unique minimal left ideal \( \mathcal{M} \) if and only if \( cr(\Omega) = 1 \) (\( \Lambda \) is a ‘perfect quasicrystal’). In this case,

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So, for perfect quasicrystals, one would like to understand the subsemigroup, \( \mathcal{J} \), of minimal idempotents.
**THEOREM** (Aujogue): If $\Lambda \subset \mathbb{R}^n$ is a codimension $k$ almost canonical cut-and-project set with hull $\Omega$ and Ellis semigroup $\mathcal{E}$ then the collection $\mathcal{J} \subset \mathcal{E}^\infty$ of idempotents is a finite sub-semigroup. For each $p \in \mathcal{J}$ there is a subtorus

$$T_p \subset \Omega_{max} \cong \mathbb{T}^{n+k}$$

so that $\mathcal{E}^\infty$ is a disjoint union of groups

$$\mathcal{E}^\infty \cong \bigcup_{p \in \mathcal{J}} \{p\} \times \tilde{T}_p$$

with coordinate-wise multiplication. Here $\tilde{T}_p := T_p + \mathbb{R}^n \subset T^{n+k}$ and if $p$ is minimal, then $\tilde{T}_p = T_p = \mathbb{T}^{n+k}$. The product in $\mathcal{J}$ is described by the face semigroup structure defined by the hyperspace arrangement determining the boundary of the window of the cut-and-project scheme.
An ‘imperfect’ quasicrystal with a non-tame Ellis semigroup:

Consider the fixed point \[01101001, 10010110, \ldots\] of the Thue-Morse substitution \[0 \mapsto 0110, 1 \mapsto 1001, \ldots\] and the corresponding bi-colored Delone set \(\Lambda\). The maximal equicontinuous factor of the hull \(\Omega = \Omega_\Lambda\) is the dyadic solenoid \(S_2\) and the coincidence rank is 2. There are two minimal left ideals, \(M^{-}\) and \(M^{+}\), in \(E = E((\Omega, \mathbb{R}))\) and \(E_\infty = M^{-} \cup M^{+}\) is a disjoint union. There are two idempotents, \(p^{\pm}\) and \(q^{\pm}\), in each minimal ideal.
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There are two minimal left ideals, \( M^- \) and \( M^+ \), in \( E = E((\Omega, \mathbb{R})) \) and

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There are two minimal left ideals, \( \mathcal{M}^- \) and \( \mathcal{M}^+ \), in \( \mathcal{E} = \mathcal{E}(\Omega, \mathbb{R}) \) and

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\mathcal{E}^\infty = \mathcal{M}^- \cup \mathcal{M}^+
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is a disjoint union.

There are two idempotents, \( p^\pm, q^\pm \in \mathcal{M}^\pm \) in each minimal ideal.
Let $\mathcal{O}$ be the (uncountable!) collection of arc-components of $S_2$. The kernels $K_f$ of

$$1 \to K_f \to f\mathcal{M} \to S_2 \to 1,$$

where $f \in \{p^\pm, q^\pm\}$ and $\mathcal{M} \in \{\mathcal{M}^\pm\}$, are each isomorphic with the group

$$\mathcal{G} := \{0, 1\}^\mathcal{O},$$

and the sequences split so that

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This gives the description:

$$\mathcal{E}^\infty \simeq \{p^\pm, q^\pm\} \times (\mathcal{G} \rtimes S_2)$$

with coordinate-wise product and the left domination rule in the first factor.