ON INDECOMPOSABILITY IN CHAOTIC ATTRACTORS

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Abstract. We exhibit a Li-Yorke chaotic interval map $F$ such that the inverse limit $X_F = \lim \{F, [0, 1]\}$ does not contain an indecomposable subcontinuum. Our result contrasts with the known property of interval maps: if $\varphi$ has positive entropy then $X_{\varphi}$ contains an indecomposable subcontinuum. Each subcontinuum of $X_F$ is homeomorphic to one of the following: an arc, or $X_F$, or a topological ray limiting to $X_F$. From a result of Barge and Martin it follows that $X_F$ is a chaotic attractor of a planar homeomorphism. In addition, $F$ can be modified to give a cofrontier that is a chaotic attractor of a planar homeomorphism but contains no indecomposable subcontinuum. Finally, $F$ can be modified, without removing or introducing new periods, to give a chaotic zero entropy interval map, such that the corresponding inverse limit contains the pseudoarc.

1. Introduction

The strong connection between dynamics of an interval map $\varphi : [0, 1] \to [0, 1]$ and topology of the inverse limit $X_\varphi = \lim \{\varphi, [0, 1]\}$ has been well documented in the last 30 years. An extensive study of this and related subjects was triggered by a series of papers by Marcy Barge and his collaborators. Among many results, Barge and Martin [3] showed that for an interval map with a periodic point of period that is not a power of 2 the inverse limit space $X_\varphi$ must contain an indecomposable subcontinuum. Barge and Martin [4] also showed that for any interval map $\varphi$ such inverse limit can be realized as an attractor of a planar homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ that restricted to $X_\varphi$ agrees with the shift homeomorphism $\sigma_\varphi$. Since then there has been a lot of attention given to the problem of relating the dynamics of a map to the topological structure of the corresponding inverse limit, and the principle that complicated dynamics induces complicated topology has become well-known and often referred to. The purpose of this article is to show that one must be careful applying this principle, as a chaotic interval map can produce a connected attractor without indecomposable subcontinua. It seems that ours is the first such example presented explicitly. This is despite the fact, that for a positive entropy map $\varphi$ the inverse limit space $X_\varphi$ must contain an indecomposable subcontinuum [30].

Theorem 1. There is a map $F : [0, 1] \to [0, 1]$ such that the inverse limit $X_F = \lim \{F, [0, 1]\}$ contains no indecomposable subcontinuum (in particular, $X_F$ is decomposable) and the induced shift homeomorphism $\sigma_F$ on $X_F$ is Li-Yorke chaotic.

The map $F$ in the above theorem can be modified to a circle map with the same properties, which by the result of Barge and Martin leads to the following theorem.

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Theorem 2. There are planar homeomorphisms $h_1$ and $h_2$, an arc-like continuum $\Lambda_1$ and cofrontier $\Lambda_2$ such that $\Lambda_1$ is a Li-Yorke chaotic attractor of $h_1$, and neither $\Lambda_i$ contains an indecomposable subcontinuum.

Before we progress, let us first briefly present definitions of some notions used above. The notion of chaos we use here comes from a paper by Li and Yorke [19]. A continuous map $\phi: X \to X$ acting on a compact metric space $(X, \rho)$ is Li-Yorke chaotic if there is an uncountable set $S \subset X$ such that $\liminf_{n \to \infty} \rho(\phi^n(x), \phi^n(y)) = 0$ and $\limsup_{n \to \infty} \rho(\phi^n(x), \phi^n(y)) > 0$ for any distinct points $x, y \in S$. It is known that there exist maps on the unit interval with zero topological entropy but Li-Yorke chaotic. These are some among the maps of type $2^\infty$, i.e. maps with points of period $2^n$ for every $n$ and no other periods.

A continuum is a nondegenerate connected and compact space. A continuum $A$ is a Li-Yorke chaotic attractor of a planar homeomorphism $h$ if $A$ is an attractor and $h|A$ is Li-Yorke chaotic. An arc-like (also snakelike, or chainable) continuum is a space that can be obtained as the inverse limit of arcs, with continuous bonding maps. Arc-like continua do not separate the plane. A cofrontier is a continuum that irreducibly separates the plane into exactly two components and is the boundary of each. A continuum is decomposable if it can be written as the union of two proper subcontinua. It is hereditarily decomposable if every subcontinuum is decomposable.

It was a long-standing conjecture of Barge that no hereditarily decomposable arc-like continuum admits homeomorphisms with positive entropy. Special case of Barge’s conjecture was proved by Ye in 1995 [30] for homeomorphisms induced by square commuting diagrams on inverse limits of arcs. Ingram [14] and Ye independently also showed that homeomorphisms of hereditarily decomposable continua admit only $2^n$-periodic orbits, so their dynamics is relatively simple. Barge’s conjecture has been recently proved by Mouron [26], and consequently hereditarily decomposable arc-like continua admit only zero entropy homeomorphisms. However, our result shows that chaotic homeomorphisms on such continua actually do exist.

The starting point of our construction is a simple, zero entropy interval map $f$ of type $2^\infty$. In Section 2, using a theorem of Bennett and Ingram [15], we are able to show that $X_f$ contains a countable family of decomposable continua, each of which is homeomorphic to $X_f$. Further, each subcontinuum of $X_f$ is a member of this family, or a topological ray limiting to such a continuum, or an arc. Next, in Section 3, we modify $f$ by a Denjoy-like construction to produce a Li-Yorke chaotic zero entropy map $F$ of type $2^\infty$. We show that this modification results in a topologically monotone factor map $\Pi: X_F \to X_f$, which guarantees that $X_F$ is hereditarily decomposable. Further, we modify $f$ to a Li-Yorke-chaotic circle map $G$ such that $X_G$ is hereditarily decomposable. The last section contains additional comments and questions related to our construction.

2. A MAP OF TYPE $2^\infty$ AND ITS INVERSE LIMIT

In this section we construct a particular example of a map of type $2^\infty$. While there are numerous different methods of construction of such a map (see, e.g. [2, 12, 24]), even of type $C^\infty$, a map $f$ considered in this section has an additional property, that its inverse limit can be easily investigated. It is the main feature demanded by us.

Define a map $f: [0, 1] \to [0, 1]$ determined by the following (see Figure 2)
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Figure 1. An hereditarily decomposable attractor $X_F$.

- $f(0) = \frac{2}{3}, f(1) = 0$,
- $f(1 - \frac{2}{3^n}) = \frac{1}{3^n}$, and $f(1 - \frac{1}{3^n}) = \frac{2}{3^n}$ for all $n \geq 1$,
- $f$ is linear between the above points.

This example was developed by Delahaye in [10] who proved that the map is of type $2^\infty$ (see also [28]).

For the reminder of this section denote by $\sigma_f$ the shift homeomorphism induced by $f$ to $X_f = \lim \{f, [0,1]\}$. For convenience, we sometimes denote $\lim \{f|_Y, Y\}$ simply by $\lim \{f, Y\}$. The projection of $X$ onto $n$-th coordinate is denoted by $\pi_n: X \ni x \mapsto x_i \in [0,1]$. Let $I_0^n = [0,1/3^n]$ for $n = 1, 2, \ldots$. These are intervals for cycles of length $2^n$, i.e. $f^{2^n}(I_0^n) = I_0^n$. Denote $I_j^n = f^j(I_0^n)$ for $j = 0, 1, \ldots, 2^n$ (we keep $I_{2^n}^n = I_0^n$ for simplicity of the notation). It can be proved that if $x \in [0,1]$ and $n > 0$ then either there is $k > 0$ such that $f^k(x) \in I_0^n$ or there is $s > 0$ such

Figure 2. Graph of $f$ and $f^2$
that $x$ is a periodic point of period $2^n$. It can also be proved that $f$ is not Li-Yorke chaotic.

Observe that $f^{2^n}|_{I^n_0}: I^n_0 \to I^n_0$ is an onto map. Denote by $X^n_0$ the inverse limit $X^n_0 = \lim_{\leftarrow} \{g_i, I^n_{-i} \pmod{2^n}\}$ where $g_i = f|I^n_{-i} \pmod{2^n}$ for $i = 1, 2, \ldots$. Denote $X^n_i = \sigma^n_f(X^n_0)$. Clearly $X^n_0$ is periodic under $\sigma_f$ and $X^n_0 = X^0_0$ and furthermore $X^n_0 + 1 \cup X^n_{2^n + 1} \subset X^n_0$.

A homeomorphic image of $[0, +\infty)$ is a topological ray and homeomorphic image of $(-\infty, +\infty)$ is a topological line.

The following useful result is attributed to Ralph Bennett. A proof (with a historical remark) can be found in [15].

**Theorem 3** (Bennett). Suppose that $g: [a, b] \to [a, b]$ is continuous and $a < d < b$ is such that $g([d, b]) \subset [d, b]$, $g|_{[a, d]}$ is monotone and there is $n > 0$ such that $g^n([d, b]) = [a, b]$. Then continuum $K = \lim_{\leftarrow} \{g, [a, b]\}$ is the union of a topological ray $R$ and a continuum $C = \lim_{\to} \{g, [a, b]\}$ such that $R \setminus R = C$.

**Lemma 4.** Each continuum $X^n_j$ is homeomorphic to $X^n_f$.

*Proof.* By induction, it is easy to see that the graph of $f^{2^n}$ on $I^n_0$ is the same as $f^{2^{n-1}}$ on $I^n_0$, that is, these maps are conjugate, or in other words, continua $X^n_0$ and $X^n_{2^n - 1}$ are homeomorphic. The theorem follows for $j \neq 0$ by the fact that, for a fixed $i$, $X^n_i = \sigma^n_f(X^n_0)$ and $\sigma_f$ is a homeomorphism.

**Lemma 5.** The continuum $X^n_f$ is the union of two continua $K_1$ and $K_2$ such that

(1) $K_1$ is homeomorphic to $K_2$,

(2) $K_1$ is the union of a topological ray $R_1$ and $X^n_0$ that compactifies $R_1$; i.e. $R_1 = X^n_0$,

(3) $K_2$ is the union of a topological ray $R_2$ and $X^n_1$ that compactifies $R_2$; i.e. $R_2 = X^n_1$,

(4) $K_1 \cap K_2 = R_1 \cap R_2 = \{\hat{p}\}$, where $\hat{p}$ is the fixed point of $\sigma_f$.

*Proof.* Let $p$ be the fixed point of $f$. Set $g = f^2$ and let $K_1 = \lim_{\leftarrow} \{g, [p, 1]\}$. Note that $g([13/21, 1]) \subset [13/21, 1]$, $g|_{[p, 13/21]}$ is monotone, and $g([p, 13/21]) = [p, 1]$. Therefore, by Theorem 3, we obtain that $K_1$ is the union of a topological ray $R_1$ and the continuum $C_1 = \lim_{\to} \{g, [13/21, 1]\}$ that compactifies $R_1$. Clearly

$$C_1 = \lim_{\leftarrow} \{g, [13/21, 1]\} = \lim_{\to} \{g, [13/2, 1]\} = X^n_3$$

and $\hat{p} = (p, p, p, \ldots)$ is the end point of $R_1$. Setting $K_2 = \lim_{\leftarrow} \{g, [0, p]\}$ the theorem follows by the fact that $\sigma_f(K_1) = K_2$.

**Corollary 6.** Each $X^n_j$ is the union of a topological line $L$ and the continua $X^n_{t+1}$ and $X^n_{t'+1}$ such that $L \setminus L = X^n_{t+1} \cup X^n_{t'+1}$, for some $t$ and $t'$.

*Proof.* This follows from the previous two lemmas.

**Theorem 7.** Continuum $X^n_f$ is hereditarily decomposible.

*Proof.* Since by Lemma 5 continuum $X^n_f$ is decomposable, we need to show that so is each subcontinuum of $X^n_f$. Let $K$ be a subcontinuum of $X^n_f$. Recall that $X^n_f$ is the union of a topological line $L$ limiting with one end to $X^n_0$ and with the other to $X^n_1$. Using the previous lemmas we will keep partitioning $X^n_f$ (if necessary) to
find where $K$ is located and realize that $K$ must be an arc, or homeomorphic to $K_1$ from Lemma 5, or homeomorphic to $X_f$. By Lemma 4 we can view each $X^n_i$ as $X_f$, in particular we can apply partitioning provided by Lemma 5 to it. We will use this fact without any further reference in the proof.

(1) suppose that $K \cap L \neq \emptyset$. If $L \subseteq K$ then $K = X_f$ and we are done. Otherwise, if $L \setminus K \neq \emptyset$, then $K$ is an arc (this is when $K \subseteq L$), or it is the union of a topological ray limiting to either $X^0_1$ or $X^1_1$, and we are done as well.

(2) suppose that $K \cap L = \emptyset$. Then either $K \subseteq X^0_1$ or $K \subseteq X^1_1$. Without loss of generality assume $K \subseteq X^0_1$.

(3) let $L_1$ be the topological line whose union with the continua $X^2_0$ and $X^2_2$, that compactify $L_1$, is $X^1_0$. In other words $\overline{L_1} \setminus L_1 = X^2_0 \cup X^2_2$ and $\overline{L_1} = X^1_0$. If $K \cap L_1 \neq \emptyset$ then we are done by the same reasoning as in (1).

(4) if $K \cap L_1 = \emptyset$ then, as in (2), we deduce that $K \subseteq X^0_1$.

(5) from the fact that $\lim_{i \to \infty} \text{diam}(X^0_1) = 0$ it follows that after finitely many steps we will be able to deduce that $K$ is an arc, or the union of a topological ray limiting to some $X^0_n$ or $K = X^1_n$ for some integers $n,j$. Namely, for $X^0_n$ such that $\text{diam}(X^0_n) < \text{diam}(K)$ we cannot have $K \subseteq X^0_n$ so the above procedure terminates.

The proof is complete. $\square$

A continuum that contains exactly $n$ topologically distinct subcontinua is called $n$-equivalent. As we exhibited in the above proof, $X_f$ is 3-equivalent. It is worth emphasizing, that an interesting example of 2-equivalent continuum was recently constructed by Islas [16], who proved that his example was hereditarily decomposable but without investigating the dynamical properties of it. In fact, Islas is using a sequence of bonding maps, so there is no easy way to induce a homeomorphism on the resulting continuum.

3. Chaos in the sense of Li and Yorke

The aim of this section is to prove Theorems 1 and 2. A starting point is the map constructed in Section 2 (recall that its graph is on Figure 2) which we consequently denote $f$.

We will perform a construction similar to that of a Denjoy map [11, Example 14.9]. First note that for all but countably many points $c \in (0,1)$ there is an open set $U \ni c$ such that $f$ is injective on $U$.

Denote by $Q$ the $\omega$-limit set of 0 under $f$ (i.e. $Q = \omega(0,f)$) and observe that for every $c \in Q$ and every $n$ there is $j$ such that $c \in I^n_j$ and hence orbit of $c$ visits each interval $I^n_j$ with period $2^n$. But diam $I^n_j = 3^{-n}$ hence the family of iterates of $f|Q$ is equicontinuous. Note that $f|Q$ is a homeomorphism, since every transitive map that has equicontinuous iterates is a homeomorphism (see [1]). It is also not hard to see that if $c \in [0,1]$ then $\omega(c,f)$ is periodic orbit (i.e. $c$ is eventually periodic) or $Q = \omega(c,f)$. Namely, if $\omega(c,f)$ in not periodic orbit then for every $n$ the orbit of $c$ has to eventually intersect the interval $I^n_0$.

Choose a point $z \in Q$, denote $D_0 = \{z, f(z)\} \cup f^{-1}(\{z\})$ and inductively $D_{n+1} = f(D_n) \cup f^{-1}(D_n)$. Finally put

\[ D_z = \bigcup_{n=1}^{\infty} D_n. \]
Since $f$ is a homeomorphism on $Q$, for points $z$ from different orbits, sets $D_z$ are disjoint. But $Q$ is uncountable and each point has finite preimage under $f$, hence we can find $z$ such that for every $c \in D_z$ there is an open set $U \ni c$ such that $f$ is an injection on $U$. Note that there at most countably many points $q \in Q$ such that $(q, q + \varepsilon) \cap Q = \emptyset$ or $(q - \varepsilon, 1) \cap Q = \emptyset$ for some $\varepsilon > 0$. Hence we may also assume that for every $\varepsilon > 0$ and for every $c \in D_z$ we have $(c - \varepsilon, c) \cap Q \neq \emptyset$ and $(c, c + \varepsilon) \cap Q \neq \emptyset$.

In particular, $D_z$ is countable and so we can enumerate its elements: assume that $D = \{y_i : i \in \mathbb{Z}\}$ where $y_i \neq y_j$ for $i \neq j$. Furthermore observe that if $f^n(y_i) = y_j$ for some $n > 0$ then $i \neq j$ and $y_j \notin \text{Orb}^+(y_j, f)$, as otherwise $z$ would be an eventually periodic point. Just by the definition, both sets $D_z$ and $[0, 1] \setminus D_z$ are invariant, i.e. $f(D_z) = D_z$ and $f([0, 1] \setminus D_z) = [0, 1] \setminus D_z$. There is also a function $\phi : \mathbb{Z} \to \mathbb{Z}$ so that $f(y_i) = y_{\phi(i)}$.

As the final step of our construction we remove all the points $y_i$ from $[0, 1]$ and fill each obtained hole with an interval $I_i$ of length $2^{-|i|}$. This way a new continuous map $F$ is defined on the extended space in such a manner that:

1. each interval $I_i$ is mapped homeomorphically onto $I_{\phi(i)}$;
2. if all intervals $I_i$ are collapsed back to single points then $F$ reverts back to the map $f$.

Condition (1) can be satisfied because the preimage $f^{-1}(y_i)$ of every $y_i$ is finite and, by the choice of $z$, the map $f$ is injective on some small neighborhood of every $y \in f^{-1}(y_i)$.

As the domain of $F$ is isometric to $[0, 4]$ we can assume that $F : [0, 4] \to [0, 4]$. In this way every interval $I_i$ becomes some interval $[a_i, b_i] \subset [0, 4]$ and there is a quotient map $\pi : [0, 4] \to [0, 1]$ that does not increase distance, collapses every interval $[a_i, b_i]$ into a single point (i.e. $\pi([a_i, b_i]) = \{y_i\}$), and has the property that $f \circ \pi = \pi \circ F$. If we fix indices $i, j \in \mathbb{Z}$, such that $y_i \notin \text{Orb}^+(y_j, f)$ then $F^n([a_j, b_j]) \cap (a_i, b_i) = \emptyset$ for all $n > 0$. This implies that there is one-to-one correspondence between periodic points of $f$ and $F$, which implies that $F$ is also of type $2^\infty$, in particular has zero topological entropy. Simply, by Misiurewicz theorem, on the interval positive entropy is equivalent to the existence of a horseshoe for some power of the map $[23]$, which easily implies existence of a periodic point with period which is not a power of 2.

In [29] Smítal characterized Li-Yorke chaos in terms of separable orbits in $\omega$-limit sets. We will use this result here. Let $\varphi : [0, 1] \to [0, 1]$ be continuous and fix two points $x_0, x_1 \in [0, 1]$. If there are two disjoint intervals $J_0, J_1$ and two integers $k_0, k_1 > 0$ such that for $i = 0, 1$ we have $x_i \in J_i$, $\varphi^{k_i}(J_i) = J_i$ and $\varphi^j(J_i)$ are pairwise disjoint for $j = 0, 1, \ldots, k_i - 1$ then we say that $x_0, x_1$ are $\varphi$-separable.

It was proved in [29, Theorem 2.2] that a map $\varphi : [0, 1] \to [0, 1]$ is Li-Yorke chaotic if and only if there is an infinite $\omega$-limit set containing two points which are not $\varphi$-separable. Note that if we fix $q \in Q \setminus D_z$ then for every $c \in D_z$ and every $\varepsilon > 0$ we have $k, s > 0$ such that $f^k(q) \in Q \cap (c - \varepsilon, c)$ and $f^s(q) \in Q \cap (c, c + \varepsilon)$. If we denote the unique point $v \in \pi^{-1}(q)$ then it is clear that $\pi^{-1}(Q \setminus D_z)$ is contained in the $\omega$-limit set of $v$ under $F$, i.e.

$$v \in \omega(v, F) \supset \pi^{-1}(Q \setminus D_z) \supset \bigcup_{i \in \mathbb{Z}} \{a_i, b_i\}.$$
Since diameters of intervals $\lim_{i \to \infty} \text{diam} I_i = 0$, there is an asymptotic (hence not $F$-separable) pair for $F$ in $\omega(v, F)$, e.g. pair $a_0, b_0$. This shows that $F$ is Li-Yorke chaotic.

Denote $X_F = \lim \{F, [0, 4]\}$. Let $\Pi: X_F \to X_f$ be given by

$$\Pi(x) = (\pi(x_1), \pi(x_2), \pi(x_3), \ldots).$$

Recall that a map $T: X_1 \to X_2$ between two continua $X_1$ and $X_2$ is (topologically) monotone if $T^{-1}(x)$ is a subcontinuum of $X_2$ for every $x \in X_1$. Equivalently, $T$ is monotone if $T^{-1}(K)$ is a subcontinuum of $X_2$ for every subcontinuum $K$ of $X_1$.

**Proposition 8.** $\Pi: X_F \to X_f$ is an onto and monotone map.

**Proof.** First note that, by definition, $\pi: [0, 4] \to [0, 1]$ is a monotone map. Now let $x \in X_F$. If $\pi_j(x) = y_j$ for some $j$ then $\Pi^{-1}(x)$ is an arc, as it is the inverse limit of $I_i$'s with the homeomorphism $F$, when restricted to either $I_i$. If $\pi_1(x) \neq y_j$ for every $j$ then $\Pi^{-1}(x)$ is a point. □

**Lemma 9.** Continuum $X_F$ is hereditarily decomposable.

**Proof.** Let $Z$ be a nondegenerate subcontinuum of $X_F$. It is enough to show that $Z$ is decomposable. Note that if $\Pi(Z)$ is a point then the projection of $Z$ from $X_F$ onto either factor space is contained in $I_j$, for some $j$. Consequently $Z$ is homeomorphic to an arc, by definition of $F$. If $\Pi(Z)$ is a nondegenerate subcontinuum of $X_f$, then $\Pi(Z) = W_1 \cup W_2$ for two proper subcontinua $W_1$ and $W_2$ of $X_f$. Since $\Pi$ is monotone we deduce that $\Pi^{-1}(W_1)$ and $\Pi^{-1}(W_2)$ are subcontinua of $X_F$ such that $Z = \Pi^{-1}(W_1) \cup \Pi^{-1}(W_2)$. This completes the proof. □

Now we are ready to prove Theorem 1.

**Proof of Theorem 1.** By Lemma 9 $X_F$ is hereditarily decomposable and by previous discussion $F$ is a continuous onto map of type $2^\infty$ which is Li-Yorke chaotic. But Li-York chaos is shared by the shift homeomorphism on inverse limits [9], hence the result follows. □

Clearly, not every map of type $2^\infty$ defines a hereditarily decomposable inverse limit. For example, when constructing the map $F$ we can define $F: I_i \to I_{\phi(i)}$ using any map fixing endpoints (e.g. maps presented in Example 4 or Example 5 in [3]), not necessarily linear homeomorphism. While such a modification has no influence on either the type of a map (new periodic points cannot be produced), or Li-Yorke chaos, an indecomposable subcontinuum such as the Knaster buckethandle continuum, or even the pseudoarc can be introduced in $X_F$.

**Remark 10.** There is a Li-Yorke chaotic interval map $\varphi$ of type $2^\infty$ such that $X_\varphi$ contains the pseudoarc.

The above observation also explains why we were so careful about the choice of the point $z$ (and the set $D_z$) for the Denjoy extension. For example $0 \in Q$ however $f^k(0)$ is a singular point (i.e. point in which $f$ is not monotone) for infinitely many values of $k > 0$. But if we insert $I_i$ in a point at which $f$ is not monotone, then $F$ must send both endpoints of $I_i$ into the same endpoint of $I_{\phi(i)}$. This forces us to send an inner point of $I_i$ into the second endpoint of $I_{\phi(i)}$, and could lead to an indecomposable subcontinuum in $X_F$.

Recall that a continuum $X$ is said to be Suslinean if every family of pairwise-disjoint subcontinua of $X$ is countable (finite or not). Note that each Suslinean
continuum is hereditarily decomposable. We note that both $X_f$ and $X_F$ are Suslinian.

**Proposition 11.** Continuum $X_f$ is Suslinian.

*Proof.* We take advantage of the partition of $X_f$ used in the proof of Theorem 7. By contradiction, suppose 8 is an uncountable cardinal and $\{C_\beta : \beta < \aleph\}$ is a family of pairwise disjoint subcontinua of $X_f$. Because the topological line limiting to the continua $X^0_0$ and $X^1_1$ is Suslinian, uncountably many $C_\beta$'s must be contained in either $X^0_0$ or $X^1_1$. Without loss of generality suppose $X^0_0$ contains uncountably many $C_\beta$'s. Since, according to Theorem 7 $X^0_0$ is a union of a topological line $L$ and two continua $X^2_0$ and $X^3_1$ homeomorphic to $X_f$, and $L$ is Suslinian, either $X^2_0$ or $X^3_1$ must contain uncountably many $C_\beta$'s. Proceeding with the continua $X^1_1$ by induction on $i$ we obtain a contradiction since otherwise for some sequence $i_n$ the set $\cap_{n=1}^{\infty} X^{i_n}$ must contain at least one continuum $C_\beta$ while it is a singleton. □

**Proposition 12.** The continuum $X_F$ is Suslinian.

*Proof.* Notice that it follows from the definition of the map $F$ that the continuum $X_F$ is obtained from $X_f$ by blow-up of some of the points to an arc. There are two types of blow-up points in $X_f$. Specifically, $f|_Q$ is a homeomorphism and there are countably many blow-up points in $Q$, hence there are also at most countably many points blown up to intervals in $\lim \{f, Q\}$. Now, let $b \in X_f \setminus \lim \{f, Q\}$ be a blow-up point. Denote $I_k = [0, 1/3^k]$ for $k = 0, 1, 2, \ldots$. First of all, since $b \notin \lim \{f, Q\}$ there exists minimal $k$ and $N > 0$ such that $b_j \notin \text{Orb}^+(I_{k+1})$ for all $j \geq N$ and if $b_j \in I_k$ then $b_j \in \text{Orb}^+(I_k)$ for all $i \geq j$. But note that if $b_j \in \text{Orb}^+(I_k) \setminus \text{Orb}^+(I_{k+1})$ for all $j \geq N$, then each $b_j$ is uniquely determined by $b_N$. It is easy to see that it is true for $\text{Orb}^+(I_0) \setminus \text{Orb}^+(I_1) = (1/3, 2/3)$ and then using mathematical induction and symmetry of the graph of $f$ we obtain (similarly to Lemma 4) that the same holds for all other $k > 0$. This shows that every $b \notin \lim \{f, Q\}$ is unique after dropping a few first positions. But then, since $\# f^{-1}(t) \leq 3$ for every $t \in [0, 1]$ on such first few coordinates and the set $D$ used in the construction of $F$ from $f$ is countable, we obtain that there are at most countably many blown up points in $X_f \setminus \lim \{f, Q\}$ (when we know $N$, there are at most countably many choices for first $N$ coordinates in each $b \notin \lim \{f, Q\}$ and then the choice for all subsequent coordinates in unique). Indeed, we have countably many blow-up points in $X_F$.

Next, suppose by the way of contradiction that $X_F$ is not Suslinian. Again, suppose 8 is an uncountable cardinal and $\{C_\beta : \beta < \aleph\}$ is a family of pairwise disjoint subcontinua of $X_f$. By Proposition 8 there is a monotone onto map $\Pi : X_F \to X_f$. Since this map is continuous the family $\{\Pi(C_\beta) : \beta < \aleph\}$ consists of compact and connected subsets of $X_f$ (some of which may be singletons). If $\Pi(C_\beta)$ is not a singleton for uncountably many $\beta$'s then we obtain a contradiction with the fact that $X_f$ is Suslinian. So $\Pi(C_\beta)$ is a singleton for uncountably many $\beta$'s. But then it follows from the definition of $\Pi$ that there would be uncountably many blow-up points in $X_f$, which is a contradiction. □

In [20] in Example 3.1 the authors provided a sequence of bonding maps $f_1, f_2, \ldots$ such that $f_n(0) = 0$ and $f_n(1) = 1$ but the inverse limit $X = \lim \{f_n\}_{n=0}^{\infty} = [0, 1]$ is not Sulinean, while is hereditarily decomposable. Hence, if we take a sequence $i_j$ such that $i_0 = 0$ and iterate backwards, so that $i_k = \phi(i_{k+1})$ then putting
(F: I_{k+1} \to I_k) = f_k \ (\text{after appropriate rescaling of domain of } f_k) \text{ we can embed } X \text{ as a subcontinuum of } X_F \text{ creating a non-Suslinean continuum.}

**Remark 13.** There is a Li-Yorke chaotic interval map \( \varphi \) of type \( 2^\infty \) such that \( X_\varphi \) is not Suslinean (but is hereditarily decomposable).

Our next objective is to prove Theorem 2.

**Lemma 14.** There is a Li-Yorke chaotic circle map \( G: S^1 \to S^1 \) such that the inverse limit \( X_G = \lim \left\{ G, S^1 \right\} \) contains no indecomposable subcontinuum.

**Proof.** Consider the map \( \bar{f}: [-1, 2] \to [-1, 2] \), a modification of the interval map \( f \) represented in Figure 3. Since \( x = -1 \) and \( x = 2 \) are fixed points of \( \bar{f} \) we can identify them to a point to obtain a circle map \( g \). It is easily checked that the inverse limit \( X_g \) is hereditarily decomposable and \( g \) can be modified again to give a Li-Yorke chaotic circle map \( G \) with \( X_G \) that contains no indecomposable subcontinuum. \( \square \)

![Figure 3. The map \( \bar{f} \).](image-url)

**Proof of Theorem 2.** The homeomorphism \( h_1 \) and the arc-like attractor \( \Lambda_1 \) exist by Theorem 1 and [4]. The homeomorphism \( h_2 \) and the cofrontier \( \Lambda_2 \) can be constructed according to [5], by the fact that \( G \) in Lemma 14 is a degree 1 circle map. \( \square \)

4. Concluding remarks

Clearly, there exist Li-Yorke chaotic maps of type \( 2^\infty \) which are \( C^\infty \)-smooth [24]. It would be interesting to know if one can improve the differentiability of our example.

**Problem 1.** Is there \( n > 0 \) such that \( \varphi \) is a \( C^n \)-smooth Li-Yorke chaotic interval map with the \( X_\varphi \) that is hereditarily decomposable? Does \( X_\varphi \) have “periodic” topological structure similar to \( X_F \) or \( X_F \) (see Lemmas 4, 5 and Figure 1)?
Also, it is known that there is an arc-like hereditarily decomposable continuum that contains no arc (e.g. see page 29 in [27]). Therefore the following question seems to be of interest.

**Problem 2.** Is there a Li-Yorke chaotic interval map \( \varphi \) such that \( \mathcal{X}_\varphi \) is hereditarily decomposable and contains no arc.

An arc-like hereditarily decomposable continuum that contains no arc should not be confused with a pseudoarc, which is hereditarily indecomposable. Recall that the pseudoarc is the unique homogeneous arc-like continuum [6],[7]. The pseudoarc contains no arc, as all subcontinua of it are indecomposable (in fact it is homeomorphic to each of its nondegenerate subcontinua). Every interval map is semi-conjugate to a pseudoarc homeomorphism [18] and the pseudoarc admits transitive homeomorphisms [17, 21]. Recently, Mouron has showed in [25] that if \( \mathcal{X}_\varphi \) is the pseudoarc then the entropy of \( \varphi \) (and the shift map \( \sigma_\varphi \)) is either 0 or \( \infty \). It is still an open question if there is a homeomorphism, or even a map, of the pseudoarc with positive finite entropy. Note that there is a zero entropy map \( \psi \) with a very simple dynamics, such that \( \mathcal{X}_\psi \) is the pseudoarc [13]. Motivated by our examples and the aforementioned results we ask the following.

**Problem 3.** Is there a Li-Yorke chaotic zero entropy homeomorphism of the pseudoarc?

At this point, it is also worth to mention that a positive answer to Problem 3 cannot be obtained using the inverse limit approach. It was proved in [8, Theorem F] that if a map \( \varphi : [0, 1] \to [0, 1] \) has a periodic point of period 2 or larger, and \( \mathcal{X}_\varphi \) is the pseudoarc, then it has a periodic point of odd period other than one. In particular, the inverse limit of a map of type \( 2^\infty \) is never the pseudoarc.

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**References**


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