Generalized pseudo-Anosov maps, inverse limits and hyperbolic 3-manifolds

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Thanks

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(Henk, Sonja and Jan: you are very nice too.)
Collaborators

This talk describes work done jointly with (not necessarily at the same time) Marcel Bertolini, Sylvain Bonnot, Philip Boyland and Toby Hall.
We describe a class of maps – *generalized pseudo-Anosov (gpA)* maps – which, as the name says, generalizes that of pseudo-Anosov maps introduced by Thurston. They come up in the study of the interface between 1- and 2-dimensional dynamics and we’ll see how they can be obtained via inverse limits. They appear naturally in parametrized families.
Introduction

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Next we discuss the family of associated mapping tori. This family contains a countable subfamily of hyperbolic 3-manifolds of finite volume, which occur when the generalized pseudo-Anosov in the family is a regular pseudo-Anosov map.
Introduction

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Next we discuss the family of associated mapping tori. This family contains a countable subfamily of hyperbolic 3-manifolds of finite volume, which occur when the generalized pseudo-Anosov in the family is a regular pseudo-Anosov map.

We then describe how inverse limits of the tent family may be viewed as laminations at the ends of a family of hyperbolic 3-manifolds provided a certain generalization of Thurston’s hyperbolization for fibered 3-manifolds holds.
Generalized pseudo-Anosov maps

A surface homeomorphism $\varphi : S \to S$ is a $gpA$ map if there exist

- a finite $\varphi$-invariant set $\Sigma \subset S$,
- a pair $(\mathcal{F}^s, \mu^s), (\mathcal{F}^u, \mu^u)$ of measured foliations with countably many singularities on $S \setminus \Sigma$ of these types

which accumulate on $\Sigma$ and nowhere else and

- a real number $\lambda > 1$

such that

$$\varphi(\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \frac{1}{\lambda} \mu^s), \quad \varphi(\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda \mu^u).$$

If $\Sigma = \emptyset$ we recover the usual definition of pA maps.
The figure shows an accumulation of singularities:
An example: the tight horseshoe

The *tight horseshoe* $\varphi$ is a ‘taut’ version of Smale’s horseshoe map. It is specially easy to describe its invariant foliations, which we do directly as follows. The map acts on the quotient space pictured below, which is a topological sphere.

$\varphi$ squeezes the square vertically, stretches it horizontally, folds it (right side folds up) and places it over itself in the horseshoe manner. The foliations of the square by horizontal and vertical segments project to the invariant foliations whose singularities are all 1-pronged at the fold points marked by dots. They all belong to a single $\varphi$-orbit which is homoclinic to a fixed ‘essential’ singularity at the lower left corner.
A second, more typical, example

Consider the self-map of the sausage-shaped disk below, which projects to the interval endomorphism shown. The disk homeomorphism is an unwrapping of the interval endomorphism as described in the Phil-Toby talks.

The shaded rectangles and half-disks contain an attracting periodic orbit and the white rectangles are squeezed vertically and stretched horizontally and mapped as shown.
The figure shows an approximation to the inverse limit of this interval endomorphism embedded in the disk. It is also the attractor of the sausage homeomorphism.
From this embedding we can construct the *stable manifolds* of the points in the inverse limit:

\[ W^s(x) = \{ y; d(\hat{f}^n(x), \hat{f}^n(y)) \to 0 \} \]

where \( \hat{f} \) is the inverse limit map acting on the disk.

They form a lamination (or foliation) transverse to the inverse limit lamination.

In order to construct a gpA map from the inverse limit we would like to collapse the gaps of this stable manifold lamination. For this it is useful to introduce the *zero-entropy equivalence relation*. 
Families

The construction of gpAs works for any map in the *tent family*, which we introduce right away. It is the family \( \{ f_\lambda \} \) of interval maps shown in the figure, parametrized by the absolute value \( \lambda \in (1, 2] \) of the slope (which is constant for each map).
Entropy

Let \((X, d)\) be a metric space and \(f : X \to X\) be a homeomorphism. A set \(E \subset X\) is \((n, \varepsilon)\)-separated if, for every \(x, y \in E\), \(d(f^i(x), f^i(y)) > \varepsilon\) for some \(0 \leq i < n\).

For a compact \(K \subset X\), let \(r(n, \varepsilon, K)\) denote the maximum number of elements in an \((n, \varepsilon)\)-separated subset of \(K\).

For compact \(K \subset X\) (not necessarily invariant) define

\[
h(f, K) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \ln r(n, \varepsilon, K)
\]

Define the entropy of \(f\) in \(K\) as \(h_{\pm}(K) = \max\{h(f, K), h(f^{-1}, K)\}\).

Finally declare two points \(x, y \in X\) to be zero-entropy equivalent if there exists a continuum \(C \ni x, y\) with \(h_{\pm}(C) = 0\).
Observe that the zero-entropy equivalence $\sim$ is $f$-equivariant so that, if $X_\sim$ denotes the quotient space, $f$ descends to a homeomorphism $f_\sim : X_\sim \to X_\sim$.

We can now state a theorem which constructs the family of *unimodal* generalized pseudo-Anosov maps.

**Theorem (Boyland-dC-Hall)**

Let $\{f_\lambda : D \to D\}$ denote the tent family embedded in the disk as a near-homeomorphism as in the example above (and as discussed in the Phil-Toby lectures). Then

- The inverse limits $\{\hat{f}_\lambda\}$ form a continuous family of disk homeomorphisms parametrized by $\lambda$.
- The zero-entropy quotients $\{(\hat{f}_\lambda)_\sim\}$ form a family of sphere homeomorphisms which are generalized pseudo-Anosov maps whenever the critical orbit of $f_\lambda$ is finite (i.e., periodic or pre-periodic).
Proof.
The first statement is the parametrized BBM construction that Phil discussed. For the second statement, we verify that the $\sim$-equivalence classes form a monotone non-separating usc decomposition of the disk. One of its elements contains the boundary, which is why we get a sphere in the quotient instead of a disk.
We believe the zero-entropy quotient family also varies continuously with $\lambda$, but we have not proved it so far. A weaker continuity statement will be discussed below.

If $\lambda$ is such that the orbit of the critical point is finite let's denote simply by $\varphi_\lambda$ the sphere homeomorphism $(\hat{f}_\lambda)_{\sim}$. These are called *unimodal generalized pseudo-Anosovs (ugpA)*.

There is a countable dense subset $\Lambda \subset (1, 2]$ of slopes for which the critical orbit of $f_\lambda$ is finite, so the family $\{\varphi_\lambda\}$ of unimodal generalized pseudo-Anosovs is naturally parametrized by $\lambda \in \Lambda$. 
The NBT family

In his thesis, Toby Hall described a family of pseudo-Anosov unimodal braids he called NBT. In the language of this talk, these correspond to parameter values $\lambda \in \Lambda$ for which the associated generalized pseudo-Anosov $\varphi_\lambda$ is in fact a regular pseudo-Anosov map, i.e., its foliations have only finitely many singularities.

Here is an example. There is a finite invariant train track in this case, which is why there are only finitely many singularities:
And here is the unstable foliation.
Let $\text{NBT} \subset \Lambda$ be the set of Hall’s parameters, i.e., the set of parameters $\lambda$ for which the $\text{ugpA}$ map $\varphi_{\lambda}$ is in fact a pseudo-Anosov map.

$\text{NBT}$ is a countably infinite subset of $\Lambda$ consisting of isolated points. It is not closed and has infinitely many accumulation points in $\Lambda$, but is far from being dense in $\Lambda$.

**Theorem (dC-Hall)**

The unimodal generalized pseudo-Anosov family is continuous on the closure $\overline{\text{NBT}}$ of the set of $\text{NBT}$ parameters.
3-manifolds

The *mapping torus* or *suspension manifold* of a surface homeomorphism $f : S \to S$ is the 3-manifold

$$M_f := (S \times [0, 1]) / \{(x, 1) \sim (f(x), 0)\}.$$ 

Mapping tori naturally fiber over the circle.
Thurston’s hyperbolization theorem for 3-manifolds which fiber over the circle states that the mapping torus of a pseudo-Anosov map admits a complete hyperbolic structure of finite volume.

It follows from the rigidity theorems of Mostow and Prasad that such manifolds are not deformable: if two of them are homeomorphic then they are equal (up to isometry).

**Conjecture**

The mapping torus of a generalized pseudo-Anosov map (which is not a pseudo-Anosov map) admits a complete hyperbolic of infinite volume. Moreover this structure is quasi-conformally rigid.

One piece of evidence for the conjecture follows.
A central theorem in Otal’s proof of Thurston’s hyperbolization for fibered 3-manifolds generalizes to the case of gpAs. A sketch of what is involved and of the statement of the theorem follows. First the cast of characters:

- \( \varphi: S \rightarrow S \) is a generalized pseudo-Anosov map, where \( S \) is a Riemann surface of finite genus with infinitely many punctures at the closure of the orbits of 1-pronged singularities of \( \varphi \).
- \( \tilde{\varphi}: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}} \) is a lift of \( \varphi \) obtained by first lifting to the upper half-plane, then extending to \( \overline{\mathbb{R}} \) (by quasi-conformality) and then reflecting to the lower half-plane.
- \( h_n: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}} \) integrates the Beltrami differential which equals the Beltrami differential of \( \tilde{\varphi}^n \) on the upper half-plane and that of \( \tilde{\varphi}^{-n} \) on the lower half-plane.
- \( \rho_0 \) is a Fuchsian representation of \( \pi_1(S) \) (compatible with the complex structure induced by \( \varphi \)) and \( \rho_n = h_n \circ \rho \circ h_n^{-1} \).
Theorem (Bertolini-dC)

If $\rho_n$ does not converge, up to conjugation, to a Kleinian representation of $\pi_1(S)$, then $\rho_n$ “converges” to an isometric action of $\pi_1(S)$ on an $\mathbb{R}$-tree.
Convergence?

Let us now go back to the ugpA family \( \{ \varphi_\lambda \}_{\lambda \in \Lambda} \). Recall that \( \text{NBT} \subset \Lambda \) is the set of parameters for which \( \varphi_\lambda \) is pseudo-Anosov. It follows from Thurston’s theorem that the mapping torus \( M_\lambda \) associated to \( \varphi_\lambda \) admits a hyperbolic structure (of finite volume) when \( \lambda \in \text{NBT} \). We also use \( M_\lambda \) to denote this structure.

As was remarked above, finite-volume hyperbolic structures on 3-manifolds are rigid so we shouldn’t expect to find non-constant continuously varying families of such manifolds. In the ugpA family a more disturbing phenomenon occurs:

**Theorem (Bonnot-dC-Hall)**

*There exist parameters \( \lambda^* \in \overline{\text{NBT}} \cap \Lambda \) for which there are different geometric limits of sequences \( M_{\lambda_n} \) for infinitely many different sequences \( \{ \lambda_n \} \subset \text{NBT} \) with \( \lambda_n \to \lambda^* \).*
In the world of infinite volume hyperbolic 3-manifolds we can hope for continuously varying non-constant families. A conjecture of sorts which would reconcile the continuity of the ugpA family with its hyperbolic structure counterpart is:

**Question**

*Assuming the ugpA mapping tori $\{M_\lambda\}_{\lambda \in \Lambda}$ admit complete hyperbolic structures, do they vary continuously with $\lambda \in \Lambda$?*
Inverse limits and hyperbolic 3-manifolds

Thurston’s theorem is proved by first constructing a hyperbolic structure on the cyclic cover $\tilde{M}_f$ of the mapping torus $M_f$ and then finding an isometric $\mathbb{Z}$-action on $\tilde{M}_f$ whose quotient is the desired hyperbolic structure on $M_f$.

Topologically, the cover $\tilde{M}_f$ is simply the product $S \times \mathbb{R}$. Geometrically, however, $\tilde{M}_f$ is very far from being a product.

One way this difference manifests itself is as follows. Since $\tilde{M}_f \cong S \times \mathbb{R}$, it has to ends, one at $+\infty$ and another at $-\infty$. There exist sequences of simple closed geodesics $\gamma_n \subset \tilde{M}_f$ whose lengths tend to 0, which exit the positive end and which, when brought back to the surface via the identification $\tilde{M}_f \cong S \times \mathbb{R}$, converge to a geodesic lamination, called the (positive) ending lamination of $\tilde{M}_f$.

This geodesic lamination is the inverse limit of the action of the map $f$ on an invariant train track.
In case the conjectures above are true and it is possible to hyperbolize the mapping tori \( \{ M_\lambda \} \) associated to the unimodal generalized pseudo-Anosovs \( \{ \varphi_\lambda \} \), we will have realized the family of inverse limits \( \{ \hat{f}_\lambda \} \) as the positive ending laminations of the cyclic covers \( \{ \tilde{M}_\lambda \} \).
The end. Thank you.