A note on minimal sets of periods for cofrontier maps

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Abstract. Using an earlier fixed-point result of the author and Nielsen-type arguments patterned on related results for torus self-maps by Alsedá at al., we describe minimal sets of periods for self-maps of cofrontiers that are inverse limits of circles. Special attention is given to the pseudocircle, ever-present in dynamical systems. Our result extends a property of circle maps, proved by Efremova, and independently by Block at al. in the late 1970s. We explain why our natural generalization is rather unexpected and shows potential of Nielsen theory for new applications.

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1. Introduction

The present paper is concerned with dynamics in circle-like cofrontiers. We prove the following new result, extending a known property of circle maps [6]; [9].

Theorem 1.1. Let \( f : \Lambda \to \Lambda \) be a degree \( d \) self-map of a circle-like cofrontier \( \Lambda \). Let

\[
\text{Per}(f) = \{ n \in \mathbb{N} : f^n(p) = p \text{ and } f^k(p) \neq p \text{ for } k < n, \text{ for some } p \in \Lambda \}. \]

1. \( \text{Per}(f) = \mathbb{N} \) for \( |d| > 2 \) or \( d = 2 \),
2. \( \mathbb{N} \setminus \{2\} \subseteq \text{Per}(f) \) if \( d = -2 \),
3. \( 1 \in \text{Per}(f) \) for \( d = 0, -1 \), and
4. if \( d = 1 \) then \( f \) may have no periodic points.

A map is a continuous function. A cofrontier \( \Lambda \) is a compact, connected set (continuum) that irreducibly separates the plane into exactly two components and is the boundary of each. It is circle-like if it can be written as the inverse limit of unit circles; i.e. \( \Lambda = \lim_{\leftarrow} \{ S^1, f_i \} \), where each \( f_i : S^1 \to S^1 \) is a map. Each subcontinuum of \( \Lambda \) is arc-like; i.e. it can be written as the inverse limit of arcs. There is a well-established interest in the behavior of dynamical systems in invariant cofrontiers, and continua in
general, as they often arise as attractors and minimal sets of such systems (see for example [2] and [16]). One cofrontier of special interest is a pseudocircle. The pseudocircle is a remarkable space first constructed by R.H. Bing in 1951 [4]. Although it shares some of the properties of the circle, with many respects the similarity becomes often very elusive. On one hand side, it is a cofrontier, has a group of rational rotations acting on it [14], and allows minimal homeomorphisms that can be extended to area-preserving planar smooth diffeomorphisms, or planar smooth diffeomorphisms having the pseudocircle as an attractor [11],[12]. It can also occur as the boundary of a Siegel disk for a holomorphic map in the complex plane [8]. On the other hand, it is not homogeneous [10],[23] (in fact neither \( \frac{1}{n} \)-homogeneous for any \( n \) [14], nor continuously homogeneous [18])); does not contain any arcs, and it is nowhere locally connected (being hereditarily indecomposable). Every proper subcontinuum of the pseudocircle is a pseudoarc. The pseudoarc, first constructed by B. Knaster in 1922 [17], is characterized as a homogenous arc-like continuum [3]. It is a simplest example of a hereditarily indecomposable continuum, and it is an object of intense study both in topology and dynamical systems [21]. Quite recently it also emerged in functional analysis [13], [22]. The pseudoarc is very important in the study of dynamics on the pseudocircle, as the arc is for dynamics on the circle. This is not only because the pseudoarc is the only proper subcontinuum of the pseudocircle, but also because it is the two-point compactification of the universal cover of the later [5]. Although there is no Nielsen theory for indecomposable continua, it is also a standard approach in Nielsen theory to investigate the properties of maps of the universal cover in order to determine some related properties of the base space. With this respect, although our main result shows a straightforward analogy between minimal sets of periods for the circle and pseudocircle, it is important to point out that the pseudoarc fails to exhibit this kind of analogy with respect to the arc. Not only it admits a transitive homeomorphism semi-conjugate to the tent map [15] (in fact every self-map of the arc is semi-conjugate to a homeomorphism of the pseudoarc [20]), but also it does not satisfy the Sharkovskii Theorem. Namely, for every \( n \) there is a homeomorphism of the pseudoarc with points of period 1 and \( n \), and no other periodic points [19]. In addition, for a circle, or interval map \( f \) it is a standard tool in dynamics to use the notion of \( f \)-covering intervals. In fact this is how the main result of the paper was proved for circle maps in [6]. This approach simply does not work for the pseudoarc or pseudocircle. One of the reasons is that for two proper subcontinua (pseudoarcs) \( P_1 \) and \( P_2 \) of either space, if \( P_1 \cap P_2 \neq \emptyset \) then \( P_1 \subset P_2 \) or \( P_2 \subset P_1 \). In addition, the pseudocircle is not a continuous image of the pseudoarc [24].

Our proof of the main result, however, does not take into consideration the topological structure of the cofrontier. Our main tool is an earlier result of the author [7] (reformulated in Theorem 2.1 below) and a Nielsen-type arguments patterned on the proof of similar results for torus maps by Alsedá, Baldwin, Llibre, Swanson, and Szlenk. It is noteworthy that the only properties that are implicitly used in [7] are connectedness, compactness, and the fact that the cofrontier is circle-like and separates the plane (note that the pseudoarc is an example of a nonseparating plane circle-like continuum).

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1 A space \( X \) is said to be homogeneous if for every \( x, y \in X \) there is a homeomorphism \( h \) such that \( h(x) = y \). \( X \) is \( \frac{1}{n} \)-homogeneous if the action of its homeomorphism group has exactly \( n \) orbits. \( X \) is said to be continuously homogeneous if for every \( x, y \in X \) there is a map \( f \) such that \( f(x) = y \).

2 A continuum is indecomposable if it can not be written as the union of two proper subcontinua, and its is hereditarily indecomposable if every proper subcontinuum is indecomposable.
To each cofrontier map $f$ we assign a Nielsen-type number that not only estimates a minimal number of fixed points, but also counts the nonempty Nielsen classes. This is done without distinguishing between essential or inessential classes, because the fixed point index is not defined on indecomposable continua. This strategy simplifies the proof to the point where only basic algebraic inequalities need to be verified. Lastly, it is important to emphasize that the goal of the paper was not to present a generalization of the Nielsen theory to new type of spaces but, besides providing a new powerful result for circle-like cofrontiers, rather to suggest the theory’s great potential for future applications to spaces that are far beyond the scope of compact polyhedra.

2. Preliminaries

Let $f : \Lambda \to \Lambda$ be a map. Note that $f$ extends to a self-map $F$ of an annulus $\mathcal{A}$, as $\Lambda$ is a closed subset of $\mathcal{A}$. We shall call $f$ a degree $d$ map, denoted by $\deg f = d$ if the degree of $F$ is $d$. This is justified by the fact that $f$ induces a multiplication by a constant $c$ on the level of the first Čech homology of $\Lambda$, that is isomorphic to $\mathbb{Z}$. If there is $F$, a degree $d$ extension of $f$, then $c = d$. We shall write $f \approx g$ if and only if $\deg f = \deg g$. Let $N$ be a Nielsen class of $F$. It is well known that $N$ is a closed and isolated subset of $\mathcal{A}$. Because $\Lambda$ is closed we can speak of a Nielsen class of $f$ given by $\bar{N} = N \cap \Lambda$. We set $n(f) = \min_{f \approx g} \# \{ \bar{N} | \bar{N} \text{ is a Nielsen class of } g, \text{ and } \bar{N} \neq \emptyset \}$.

The following theorem is a reformulation of the result proved by the present author in [7].

**Theorem 2.1.** Let $f : \Lambda \to \Lambda$ be a degree $d \neq 1$ map. Then $n(f^n) = |d^n - 1|$.

The following lemma is a modification of Lemma 2.1 in [1] to the case of self-maps of $\Lambda$.

**Lemma 2.2.** Let $f : \Lambda \to \Lambda$ be a map. Each fixed point class of $f^k$ is contained in a fixed point class of $f^n$ if $k$ divides $n$.

**Proof.** The lemma holds for self-maps of an annulus, as a special case of torus maps. Let $F : \mathcal{A} \to \mathcal{A}$ be an extension of $f$. Any fixed point class of $F^k$ is of the form $\bar{N} = N \cap \Lambda$, for a fixed point class $N$ of $F^k$. By Lemma 2.1 in [1], if $k$ divides $n$, $N$ is contained in a fixed point class $K$ of $F^n$. But $K = K \cap \Lambda$ is a fixed point class of $f^n$ so $\bar{N} \subset K$. □

Our next Lemma is an adjustment of Proposition 2.2 in [1].

**Lemma 2.3.** Assume that $n(f^n) > \sum_{\# \text{prime}} n(f^k)$

Then $f$ has a periodic point of period $n$.

**Proof.** Let $d = \deg(f)$. Without loss of generality we assume $d \neq 1, -1, 0$. We will use arguments presented in [1], with an adjustment for a new meaning of $n(f)$. Let $C_1, \ldots, C_{[d^n-1]}$ be the Nielsen classes of $f^n$. Recall that each $C_i$ intersects $\Lambda$ in a nonempty set. For each $i$, choose $x_i \in C_i$ and by contradiction suppose $f^{k_i}(x_i) = x_i$ is with $k_i \neq n$. $k_i$ can be chosen so that $\frac{n}{k_i}$ is prime. Denote by $\bar{C}_i$ the Nielsen class
of \( f^{k_i} \) that contains \( x_i \). By Lemma 2.2 \( C_i \subset C_i \). But this gives a contradiction since \( n(f^n) > \sum_{p \text{ prime}} n(f^p) \) and \( C_i \)'s are pairwise disjoint.

\( \square \)

3. Proof of Theorem 1.1

Proof. Note that (3) is a special case of Theorem 2.1. (4) holds for an irrational rotation of a circle, and for the pseudocircle it follows from [11]. Now let \( f : \Lambda \to \Lambda \) be a degree \( d \) map, where \(|d| > 1\).

First suppose \( d > 1 \). Note that \( n(f^k) = d^k - 1 \) for every integer \( k > 1 \). Therefore for every integer \( n > 0 \) we have

\[
n(f^n) = d^n - 1 = (d - 1)(d^{n-1} + d^{n-2} + \ldots + 1) > \sum_{1 \leq k \leq n-1} n(f^k).
\]  

Consequently \( n(f^n) > \sum_{p \text{ prime}} n(f^p) \) and by Lemma 2 \( f \) has a periodic point of period \( n \) for every integer \( n > 0 \).

Second consider a map of degree \(-d\) with \( d > 1 \). Note that for \( k \) even we have \( n(f^k) = d^k - 1 \), whereas for an odd \( k \) we get \( n(f^k) = d^k + 1 > d^k - 1 \) and using (3.1) and Lemma 2 we deduce again that \( f \) has a periodic point of period \( n \) for every integer \( n > 0 \). If \( n \) is even then

\[
n(f^n) = (d - 1)(d^{n-1} + d^{n-2} + \ldots + 1) = (d - 1) \sum_{1 \leq k \leq n-1} n(f^k),
\]  

and therefore \( n(f^n) > \sum_{p \text{ prime}} n(f^p) \) for any \(-d < -2\). Since \( \sum_{1 \leq k \leq n-1} n(f^k) = \sum_{p \text{ prime}} n(f^p) \) only for \( n = 2 \) we separately check that \( n(f^2) = n(f) = 3 \) for \(-d = -2\). Therefore \( f \) of degree \(-2\) has a periodic point of period \( n \) for every integer \( n > 2 \) and \( n = 1 \).

\( \square \)

References


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