Finite-sheeted covering spaces and a near local homeomorphism property for pseudosolenoids

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Abstract. Fox theory of overlays is used to characterize finite-sheeted covering spaces of pseudosolenoids (i.e. hereditarily indecomposable circle-like nonchainable continua). Let $S$ be a $P$-adic pseudosolenoid, for a sequence of prime numbers $P = (p_1, p_2, \ldots, p_n, \ldots)$. $S$ admits a $d$-fold cover onto itself if and only if $d$ is relatively prime to all but finitely many $p_i$. Moreover, if $C$ is any connected finite-sheeted covering space of $S$ then $C$ is homeomorphic to $S$. These results parallel known results for solenoids and extend results of Bellamy and Heath for the pseudocircle. In addition, it is shown that most self-maps of pseudosolenoids are finite-sheeted covering maps. Namely, if $E(S)$ is the space of all surjective self-maps of $S$ with the sup metric, then the subset $C(S)$ of local self-homeomorphisms of $S$ is a dense $G_δ$ in $E(S)$. This result, based on a theorem of Kawamura, extends the near-homeomorphism property of the pseudoarc proved independently by Lewis and Smith, and relates to a question raised by Lewis in 1984, as to which other nondegenerate continua have the near-homeomorphism property.


Keywords. pseudocircle, pseudosolenoid, covering space.

1. Introduction

This paper concerns finite-sheeted covering spaces of pseudosolenoids and is motivated by a series of papers that accumulated on the subject of finite-sheeted covering spaces of solenoids in recent years [7],[15],[19] (see also [1]). A continuum is a connected and compact metric space that contains at least two points. A pseudosolenoid is any hereditarily indecomposable circle-like nonchainable continuum. Pseudosolenoids were characterized by L. Fearnley [10] and J. T. Rogers Jr. [32].
According to the classification there is only one topologically distinct planar continuum in this class. Namely, every such planar continuum is homeomorphic to the pseudocircle described by R. H. Bing [4] in 1951. Pseudosolenoids were classified by Fearnley and Rogers [10], [32]. According to their classification, there are uncountably many topologically distinct pseudosolenoids. Two pseudosolenoids are homeomorphic if and only if they have isomorphic first Čech cohomology groups. Equivalently, two pseudosolenoids are homeomorphic if and only if each of them can be mapped onto the other. Unlike the pseudoarc, which is homogeneous [5], the pseudocircle and nonplanar pseudosolenoids are nonhomogeneous [11], [31]. Pseudosolenoids have been extensively studied in the recent 60 years, and the interest in these peculiar spaces was amplified by the examples from dynamical systems (see e.g. [8], [16] and [22]). There is also a quite recent surprising connection to functional analysis through the pseudoarc [21], [30]. However, with the only exception for the results of Jo Heath [18], and David Bellamy [2] for the pseudocircle, there does not appear to be anything in the literature regarding the \(d\)-fold covering spaces of pseudosolenoids. Heath showed that the pseudocircle admits a 2-fold cover onto itself and the generalization to any \(d > 1\) follows from the result of Bellamy (see also [14] and [17]). However their results do not determine if every finite-sheeted connected covering space of the pseudocircle must be homeomorphic to it. In addition, every nonplanar pseudosolenoid can be mapped onto the pseudocircle [11], [31]. On the other hand it is well known that, for a sequence of prime numbers \(P = (p_1, p_2, \ldots, p_n, \ldots)\), the \(P\)-adic solenoid \(S_P\) admits a \(d\)-fold cover onto itself if and only if \(d\) is coprime with all but finitely many \(p_i\) (see [15] and [19] for related recent results), and that every such connected covering space must be homeomorphic to \(S_P\). Although solenoids are homogeneous, contrasting with nonhomogeneity of pseudosolenoids, there is a natural connection between solenoids and pseudosolenoids. Namely, the two classes of continua are in a bijective correspondence that assigns to each solenoid a pseudosolenoid that has an isomorphic Čech cohomology group. This motivated writing of the present paper in which using Fox theory of overlays, we shall provide the following characterization.

**Theorem 1.1.** Let \(P = (p_1, p_2, \ldots, p_n, \ldots)\) be a sequence of primes and let \(S\) be a \(P\)-adic pseudosolenoid.

1. \(S\) admits a \(d\)-fold cover onto itself if and only if \(d\) is relatively prime to all but finitely many \(p_i \in P\).

2. If \(C\) is a continuum and \(\tau : C \to S\) is a \(d\)-fold covering map then \(C\) is homeomorphic to \(S\).

In [26] W. Lewis and in [35] M. Smith independently showed that every surjective self-map of the pseudoarc \(P\) is a near-homeomorphism; i.e. for any map \(f : P \to P\) and any \(\epsilon > 0\) there is a homeomorphism \(h : P \to P\) such that \(\sup\{|f(x) - h(x)| : x \in P\} < \epsilon\). Lewis also asked which other nondegenerate

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1The reader interested in learning about the universal covering space of the pseudocircle should consult [3], [14] and [24].
continua \( X \) (besides the pseudoarc) have the property that the space of homeomorphisms of \( X \) is a dense \( G_δ \) in the space of self-maps of \( X \). Although the pseudosolenoids do not provide an answer to the question, admitting finite-sheeted covering maps, they do satisfy a slightly weaker property. Namely, we shall show that a surjective self-map of a pseudosolenoid is arbitrarily close to a local homeomorphism.

**Theorem 1.2.** [Near Local Homeomorphism Property] If \( E(S) \) is the space of all surjective self-maps of the pseudosolenoid \( S \) with the sup metric, then the subset \( C(S) \) of local self-homeomorphisms of \( S \) is a dense \( G_δ \) in \( E(S) \).

Our proof of the above result will heavily rely on a result of Kazuhiro Kawanmura, who in [20] showed that every surjective self-map of a pseudosolenoid is a near-homeomorphism if and only if it induces an automorphism of the first Čech cohomology. In the following example we note that the solenoids do not have the near local homeomorphism property. Although this fact may follow from some more general results on solenoids, the purpose of the example is to give the reader more insight into the near local homeomorphism property by showing the kind of mapping property that cannot occur on a pseudosolenoid.

**Example 1.** Fix \( a \in [0, \frac{1}{2}] \) and consider the piecewise linear map on the unit interval \([0,1]\) determined by the following conditions \( f_a(0) = 0, f_a(\frac{1}{2} - a^2) = \frac{1}{2} + a^2, f_a(\frac{1}{2} + a^2) = \frac{1}{2} - a^2, f_a(1) = 1 \). For \( a \in [\frac{1}{2}, 1] \) define \( f_a = f_{1-a} \). Notice that \( f_0(x) = f_1(x) = x \) for every \( x \), and \( f_a(x) \) is not a homeomorphism when restricted to any neighborhood of \( \frac{1}{2} - a^2 \) or \( \frac{1}{2} + a^2 \) for any \( a \neq 0, 1 \). It is easily observed that if \( h : [0,1] \to [0,1] \) is a homeomorphism then \( |f_{\frac{1}{2}} - h| = \sup\{|f_{\frac{1}{2}}(x) - h(x)| : x \in [0,1]\} \geq \frac{1}{2} \).

Now let \( S_P \) be a solenoid. Choose a point \( g \in S_P \) and let \( U \) be a small open neighborhood of \( g \), so that the closure \( \text{cl}(U) \) of \( U \) is homeomorphic to \( K \times [0,1] \), where \( K \) is a Cantor set. Now define \( g : S_P \to S_P \) by \( g(z) = g(a,t) = f_a(t) \) for any \( z = (a,t) \in K \times [0,1] \) and \( g(z) = z \) if \( z \notin \text{cl}(U) \). Without loss of generality we may assume that the metric on \( S \), when restricted to \( \text{cl}(U) \), agrees with the maximum metric\(^2\) on the product \( K \times [0,1] \). By the fact that \( \text{dist}(f,g) \geq \frac{1}{2} \) for any homeomorphism \( h \) of \([0,1]\) the map \( g \) cannot be approximated by a local homeomorphism of \( S_P \) for \( \epsilon < \frac{1}{2} \).

Let us also point out that for a given local homeomorphism \( t : S_P \to S_P \) and \( \epsilon > 0 \), by an analogous construction, one can obtain a map \( t' : S_P \to S_P \) such that \( |t - t'| < \epsilon \) and \( t' \) is not a local homeomorphism. It is simply enough to compose \( t \) with a map similar to \( g \), that is \( \epsilon \)-close to the identity on \( S_P \). Therefore the set of maps that are not local homeomorphisms is dense in the set of all self-maps of \( S_P \). This shows a very strong negation of Theorem 1.2 in the case of solenoids\(^3\).

\(^2\)The maximum metric \( d : [0,1] \times [0,1] \to [0,1] \) is defined by \( d((x_1, x_2), (y_1, y_2)) = \max\{|x_1 - y_1|, |x_2 - y_2|\} \).

\(^3\)We are grateful to an anonymous referee who brought this fact to our attention.
2. Preliminaries

All spaces considered will be metric. For any of them we will always denote the distance between two elements $x$ and $y$ by $|x - y|$. Recall that a continuum $X$ is indecomposable if it is not the union of two proper subcontinua. A continuum is hereditarily indecomposable if every subcontinuum is also indecomposable. An arc is $\epsilon$-crooked if for each pair of its points $p$ and $q$ there are points $r$ and $s$ between $p$ and $q$ on the arc such that $r$ lies between $p$ and $s$, $|p - s| < \epsilon$, and $|r - q| < \epsilon$. Bing proved that $\epsilon$-crookedness characterizes hereditarily indecomposable continua. In the realm of finite-dimensional spaces it means that the continuum $X$ is hereditarily indecomposable if and only if it is the intersection of a monotone decreasing (in the sense of inclusion) sequence $\{S_i : i \in \mathbb{N}\}$ of polyhedra, such that each arc in $S_i$ is $\frac{1}{i}$-crooked [6]. An open cover $U$ of a continuum $X$ is called a chain (circular chain) if the nerve of $U$ is homeomorphic to an arc (circle). The elements of $U$ are called links. A $k$-fold covering map of the space $Y$ is a surjection $\tau : X \to Y$ with the property that for every $y \in Y$ there is an open neighborhood $W$ of $y$ such that $\tau^{-1}(W)$ consists of $k$ pairwise disjoint sets, each of which is mapped by $\tau$ homeomorphically onto $W$. In such a case $X$ is called a $k$-fold covering space. Note that, although in general the notion of a covering space is broader than the one of an overlay structure, $X$ is a $k$-fold covering space of $Y$ if and only if $X$ is an overlay of $Y$ by Proposition 2.2 in [28]. This will enable us to take advantage of the Fox theory of overlays in [13]. Fox theory was a successful effort to extend the classical theory of covering spaces for connected, locally connected, semi-locally 1-connected spaces to arbitrary metrizable spaces. As is well known, the fundamental theorem of the covering space theory asserts that the connected $d$-fold covering spaces of a base space $Y$ are in a bijective correspondence with the homotopy classes of transitive representations of the fundamental group of $Y$ in the symmetric group of degree $d$. Fox achieved an analog of this theorem ([13], Theorem 6.1] replacing the notion of a fundamental group by its shape theoretic version, namely the fundamental pro-group. Intuitively, for a continuum $Y$ given

![Diagram](image-url)
by an inverse limit of polyhedra one considers the corresponding inverse system of fundamental groups that, up to an equivalence, determines the fundamental pro-group of \( Y \). Among other results, Fox also showed \([13],\) Extension Theorem that an overlay of a subspace \( X \) of a separable metric space, extends to an overlay of an open neighborhood of \( Y \). These two results will be helpful in the sequel, in proving Theorem 1.1. For more details of Fox theory we refer reader to \([13],\) as well as to its more recent discussion and extension to arbitrary topological spaces in \([27]\).

For the remainder of the paper \( P = (p_n : n \in \mathbb{N}) \) will be a sequence of prime numbers and \( S_P \) will be the \( P \)-adic solenoid; i.e. \( S_P \) is homeomorphic to the inverse limit of unit circles \( S^1 \) in the complex plane, centered at 0, with bonding maps \( f_n \) given by \( f_n(z) = z^{p_n} \). Let \( S \) denote a pseudosolenoid with the \( Čech \) cohomology group isomorphic to that of \( S_P \). We shall consider \( S_P \) and \( S \) as subspaces of a solid torus \( T \). \( S \) is the intersection of a descending family of solid tori \( \{ T_n : n \in \mathbb{N} \} \), such that \( T_{n+1} \) is embedded into \( T_n \) with degree \( p_n \) and every arc in \( T_n \) is \( \frac{1}{n} \)-crooked. Note that each \( T_n \) is homotopic in \( T \) to a torus \( R_n \) such that the embedding \( e_n : R_n \to R_{n-1} \) is monotone with respect to the angular coordinate, as each degree \( p_n \) map of \( S^1 \) is homotopic to \( f_n \). In particular \( \cap_{n \in \mathbb{N}} R_n \) is homeomorphic to \( S_P \). Since by Fearnley’s and Rogers’ characterization each pseudosolenoid is characterized by its first \( Čech \) cohomology group \([10], \) Theorem 3.3 and \([32], \) Theorem 8, there is a lot of freedom in the construction of \( S \). Consequently, one can think of each \( T_n \) as a union of convex 3-dimensional balls (convex balls can be used as links of the circular chains defining \( S \)). The following lemma is motivated by Example 1 in \([18]\), where Lemma 1 (4) \([18]\) was used to prove hereditary indecomposability of the covering space. Because this lemma is explicitly proved in \([17]\) (Corollary 3) only in the case of a 2-fold covering map, we use independent arguments.

**Lemma 2.1.** If \( C \) is a continuum and \( \tau : C \to S \) is a finite-sheeted covering map of the pseudosolenoid \( S \) then \( C \) is a pseudosolenoid.

**Proof.** We need to show that \( C \) is circle-like and hereditarily indecomposable (observe that this will imply that \( C \) is nonchainable, as no pseudosolenoid is a continuous image of the pseudoarc). This is achieved in the three following claims. \( \square \)

**Claim 2.2.** \( C \) is circle-like.

**Proof.** (of Claim 2.2) Since \( \tau : C \to S \) is a finite-sheeted covering map and since \( C \) is compact, there is an \( \epsilon_0 > 0 \) such that \( \tau \) restricted to any \( \epsilon_0 \)-ball is a homeomorphism from such a ball to its image. Fix \( \epsilon < \epsilon_0 \) and choose \( \delta \) small enough so that each copy of \( B \) in \( \tau^{-1}(B) \) has diameter less than \( \epsilon/10 \) for any \( \delta \)-ball \( B \). Now if \( U \) is an open cover of \( S \) by \( \delta \)-balls with the nerve of \( U \) homeomorphic to \( S^1 \) then each vertex of the nerve is of degree 2. For \( U \in \mathcal{U} \) let \( \mathcal{V}_U \) be the collection of the homeomorphic copies (sheets) of \( U \) in \( \tau^{-1}(U) \). It follows that \( \mathcal{V} = \bigcup_{U \in \mathcal{U}} \mathcal{V}_U \) is an open \( \epsilon \)-cover of \( C \) and each vertex of the nerve of \( \mathcal{V} \) is of degree 2. Moreover, \( C \) is connected and each element \( V \) intersects \( C \), so \( V \) is a circular \( \epsilon \)-chain. Consequently \( C \) is circle-like. \( \square \)
Claim 2.3. \( \tau \) extends to a finite-sheeted covering map \( \phi \) between two tori.

Proof. (of Claim 2.3) By the Extension Theorem on p. 60 in [13] any covering map onto \( S \) is a restriction of a covering map between two tori. Namely, there is an embedding of \( S \) in an open torus \( \bar{U} \), an embedding of \( C \) into an open torus \( \bar{U} \) and a \( d \)-fold covering map \( \phi : \bar{U} \to U \) such that \( \phi(C) = \tau \) and \( \phi^{-1}(S) = C \). To achieve such an embedding of \( C \) one lets \( \cU \) be a circular chain of open convex 3-dimensional balls covering \( S \) such that \( \cU = \bigcup \cU \) is an open torus. Then, by application of the proof of the Extension Theorem (p. 61 in [13]), there is an open covering \( \bar{\cU} \) of \( C \) such that each element of \( \bar{\cU} \) is homeomorphic to an element of \( \cU \) and a covering map \( \phi : \bar{\cU} \to \cU \) with \( \phi|C = \tau \), where \( \bar{\cU} = \bigcup \bar{\cU} \) is an open torus since it is a finite-sheeted covering space for \( \cU \).

Claim 2.4. \( C \) is hereditarily indecomposable.

Proof. (of Claim 2.4) Let \( Z \) be a proper subcontinuum of \( C \). Then \( Z \) is chainable since \( C \) is circle-like, and so \( Z \) is the image of a pseudoarc by Corollary 3 in [25] (or Theorem 3.5 in [12]). Therefore \( \tau(Z) \) cannot be \( S \), since the pseudoarc cannot be mapped onto any pseudosolenoid by [10] Theorem 4.3 (see also [33]). Now since \( \tau(Z) \) is a proper subcontinuum of \( S \) it is a pseudoarc. By Claim 2.3, \( \tau \) extends to a finite-sheeted covering map between tori and since \( \tau(Z) \) is a pseudoarc (and thus acyclic), it will have a simply connected neighborhood in the torus containing \( S \) that will lift to homeomorphic copies in the toroidal covering space containing \( C \), one of which contains \( Z \). Thus \( Z \) and \( \tau(Z) \) are homeomorphic and consequently every proper subcontinuum of \( C \) is homeomorphic to a pseudoarc, and therefore indecomposable. Now by contradiction suppose that \( C \) is decomposable\(^4\). Let \( A \) and \( B \) be two proper subcontinua such that \( C = A \cup B \) and let \( x \in A \cap B \). Since each composant of a continuum is dense, there are two proper subcontinua \( A' \) and \( B' \) of \( A \) and \( B \) respectively such that \( x \in A' \cap B', A' - B \neq \emptyset \), and \( B' - A \neq \emptyset \). Consequently \( A' \cup B' \) is a proper decomposable subcontinuum of \( C \) leading to a contradiction.

To prove that the pseudosolenoids have the Near Local Homeomorphism Property the crucial tool will be the following result of K. Kawamura. Note that it follows from Kawamura’s result and Theorem 1.1 that there are surjective self-maps of \( S \) that are not near-homeomorphisms.

Theorem 2.5. [20] If \( f : S \to S \) is a surjective map of a pseudosolenoid then it is a near-homeomorphism if and only if it induces an isomorphism of the first Čech cohomology of \( S \).

Kawamura’s result depends deeply on the properties of crooked circular chains defining pseudosolenoids. However, we will not need to analyze them in

\(^4\)It seems it is well known that if each proper subcontinuum of a continuum \( X \) is indecomposable then \( X \) is indecomposable. Since we were unable to locate a reference for this result, we included a short explanation for completeness sake.
order to prove Theorem 1.2. Instead we shall observe that maps that are not arbi-
trarily close to a homeomorphism are compositions of near-homeomorphisms with
covering maps.

3. Proof of Theorem 1.1 and 1.2

Proof. (of Theorem 1.1) (1) Consider \( S \) as a subset of a solid torus \( T \) and let
\( \tau : T \to T \) be a \( d \)-fold covering map. Note that by Lemma 2.1 each component of
\( \tau^{-1}(S) \) is a pseudosolenoid. Let \( T_n \), for \( n \in \mathbb{N} \), be a sequence of solid tori, such that
\( T_{n+1} \) is embedded in \( T_n \) with degree \( p_i \) and such that \( S = \bigcap_{n \in \mathbb{N}} T_n \), as described
in the Preliminaries. Since the fundamental trope (the first homotopy pro-group) of \( S \) is the same as the fundamental trope of \( S_P \), and is given by the sequence
\[
\mathbb{Z}(1) \xleftarrow{p_1} \mathbb{Z}(2) \xleftarrow{p_2} \mathbb{Z}(3) \xleftarrow{p_3} \ldots
\]
by Example 2 in [13] we get that \( \tau^{-1}(S) \) is connected if and only if \( d \) is coprime
with all but finitely many \( p_i \), and if \( \tau^{-1}(S) \) is connected then it is unique (see also [13], Theorem 6.1). This is because the fundamental trope of \( S \) does not recognize
whether \( T_n \) contains only \( \frac{1}{n} \)-crooked arcs or not, but only depends on the fact that
for each \( n \) there is a homotopy \( h_n \) between \( R_n \) and \( T_n \) in \( T \), where \( S_P = \bigcap_{n \in \mathbb{N}} R_n \).

Now suppose \( d \) is coprime with all but finitely many \( p_i \). Because the unique
connected \( d \)-fold covering space of \( S_P \) is homeomorphic to \( S_P \) ([15], Theorem 2), it follows that \( \tau^{-1}(S) \) and \( S \) are shape equivalent by Theorem 5 [34], and consequently \( \tau^{-1}(S) \) is homeomorphic to \( S \), by the characterization in [10] and
[32].

(2) Let \( C \) be a continuum and \( \tau : C \to S \) be a \( d \)-fold covering map. \( \tau \) extends
to a \( d \)-fold covering map \( \phi \) between two tori. By Lemma 2.1 \( C \) is a pseudosolenoid.
As noted in (1) it follows that \( d \) must be coprime with all but finitely many \( p_i \), \( C \)
is unique and \( C \) is homeomorphic to \( S \). \( \square \)

Now we are going to prove that every pseudosolenoid has the Near Local
Homeomorphism Property. For a surjective self-map \( f \) of \( S \) the approximation by
a local homeomorphism is obtained by lifting \( f \) to a shape equivalence, approx-
imating it by a homeomorphism and then composing the latter with a covering
map.

Proof. (of Theorem 1.2) Fix \( \epsilon > 0 \). Let \( S \subseteq T \) be the \( P \)-adic pseudosolenoid
embedded into the torus \( T \), as described earlier. \( f \) extends to a map \( F : T_0 \to T \),
defined on a toroidal neighborhood \( T_0 \subseteq T \), as \( S \) is a closed subset of \( T \) (the torus
\( T_0 \) is embedded essentially into \( T \), perhaps with a degree different from 1). Notice
that \( F(S) = S \). Let \( T_0 \supseteq T_1 \supseteq T_2 \supseteq \ldots \) be a sequence of tori such that \( S \) is
the intersection of that sequence. One can find, for each \( n \), a torus \( R_n \) such that
\( F(T_n) \subseteq R_n \) and such that \( \bigcap_{n \in \mathbb{N}} R_n = F(S) = S \).
The first cohomology groups $H^1(T_n)$ and $H^1(R_n)$ are isomorphic to $\mathbb{Z}$. $F$ induces multiplication by an integer $d$ between $H^1(T_n)$ and $H^1(R_n)$. It is a consequence of Theorem 5 in [33] that $d \neq 0$. $H^1(S)$, the first Čech cohomology group of $S$, is isomorphic to the direct limit of $\{H^1(T_n) : n \in \mathbb{N}\}$ (see [29], Theorem 73.4, p.440). $\hat{H}^1(F(S))$ is isomorphic to the direct limit of $\{H^1(R_n) : n \in \mathbb{N}\}$. If $d = \pm 1$ then $f$ induces an isomorphism between $H^1(T_n)$ and $H^1(R_n)$ for each $n$, and therefore between $H^1(S)$ and $H^1(F(S))$. Consequently Kawamura’s result applies.

Suppose $d \neq \pm 1$. Consider a $d$-fold covering $\phi : T_d \to \mathbb{T}$. As $\phi_\#(\pi_1(T_d)) = F_\#(\pi_1(T_0))$ there is a lift $\tilde{F} : \mathbb{T} \to T_d$ of $F$. There is a component $S_d$ of $\phi^{-1}(S)$ such that $\tilde{F}(S) = S_d$. By Lemma 2.1 $S_d$ is a pseudosolenoid. Since both $S$ and $S_d$ are continuous images of each other, by Theorem 4.3 in [10], $S$ and $S_d$ are homeomorphic. Set $\tilde{F} : S = \tilde{F} \circ g : S \to S_d$. Notice that $\tilde{F}$ induces multiplication by $\pm 1$ between $H^1(\mathbb{T})$ and $H^1(T_d)$, as $\phi$ induces multiplication by $d$ between $H^1(T_d)$ and $H^1(T)$. Consequently, $\tilde{F}$ induces an isomorphism between the Čech cohomology of $S$ and $S_d$. Pick $\delta > 0$ so that the image of any $\delta$-ball in $T_d$ by the mapping $p$ has diameter less than $\epsilon$, and so that for every $x \in S$ there is a $\delta$-ball $D$ around $x$ such that $\phi|D$ is a homeomorphism onto $\phi(D)$ (such a $\delta$ exists as $\phi$ is a covering map). By Kawamura’s result, there is a homeomorphism $h : S \to S_d$ such that $|\tilde{f} - h| < \delta$, $p \circ h : S \to S$ is the desired local homeomorphism.

Finally, let $E(S)$ be the space of all surjective self-maps of $S$ with the sup metric, and $C(S)$ the subset of local self-homeomorphisms of $S$. Since for a fixed $\epsilon > 0$ and $f \in E(S)$ there is an $h \in C(S)$ that is at most $\epsilon$ away from $f$, it follows that $C(S)$ is dense in $E(S)$. For every $d > 1$ relatively prime with all but finitely many $p_i$’s in $P$ let $\tau_d$ be the $d$-fold covering map of $S$ induced by the $d$-fold covering map of the torus. Fix integers $i > 0$ and $d$. Let

$$M^d_{\tau_d}(S) = \{\tau_d \circ g : g \in E(S), \text{diam}(g^{-1}(x)) < \frac{1}{i} \text{ for all } x \in S\}.$$ 

Let $\Gamma_d : E(S) \to E(S)$ be a transformation given by $\Gamma_d(g) = \tau_d \circ g$ for every $g \in E(S)$.

**Claim 3.1.** $\Gamma_d$ is an open mapping.

**Proof.** (of Claim 3.1) Fix $\epsilon > 0$. There is a $\delta > 0$ such that $|\tau_d(x) - \tau_d(y)| < \epsilon$ for $|x - y| < \delta$. Consequently if $|f - g| < \delta$ then $|\Gamma_d(f) - \Gamma_d(g)| < \epsilon$. This proves continuity of $\Gamma_d$. Analogously, fix $\delta > 0$ and note that there is an $\epsilon > 0$ such that $|\tau_d(x) - \tau_d(y)| < \epsilon$ implies $|x - y| < \delta$, because $\tau_d$ is open. Consequently if $|\Gamma_d(f) - \Gamma_d(g)| < \epsilon$ then $|f - g| < \delta$. \qed

**Claim 3.2.** Each $M^d_{\tau_d}(S)$ is open in $E(S)$.

**Proof.** (of Claim 3.2) Note that for every $i > 0$ the set $M_i = \{g \in E(S) : \text{diam}(g^{-1}(x)) < \frac{1}{i} \text{ for all } x \in S\}$ is open in $E(S)$ ([26], Lemma 1.3). Since $M^d_{\tau_d}(S) = \Gamma_d(M_i(S))$, by Claim 3.1 we conclude that $M^d_{\tau_d}(S)$ is open. \qed
It follows that $G_i = \bigcup_{d \in \text{Rel}(P)} M^d_i(S)$ is open, where $\text{Rel}(P)$ is the set of all natural numbers relatively prime to all but finitely many elements of $P$. Since $\bigcap_{i \in \mathbb{N}} G_i = C(S)$ it follows that $C(S)$ is a dense $G_\delta$ in $E(S)$.

\[ \square \]

4. Conclusions

We conclude by raising some questions, in the spirit of existing results on the pseudoarc and solenoids.

**Question 1.** Is there another continuum $X$ that has the Near Local Homeomorphism Property and not the Near Homeomorphism Property? Must $X$ be hereditarily indecomposable?

In [26] Lewis showed that any self-map of a chainable continuum can be lifted to a self-homeomorphism of the pseudoarc. It seems reasonable to expect that, under some assumptions, a similar result may hold for circle-like continua.

**Question 2.** Under what assumptions can a surjective self-map of a circle-like continuum be lifted to a local self-homeomorphism of a pseudosolenoid; i.e. if $f : X \to X$ is a self-map of a circle-like continuum $X$, is there a pseudosolenoid $S$, a surjection $p : S \to X$ and a local homeomorphism $\tau : S \to S$ such that $f \circ p = p \circ \tau$?

A further generalization of the near homeomorphism property is developed in [36], and explored in the context of the pseudoarc. A natural question to ask is the following.

**Question 3.** For a given pseudosolenoid, $S$, can each subcontinuum of $S \times S$ that projects onto both coordinate spaces be approximated (in the hyperspace of $S \times S$) by a local homeomorphism.

Finally, in [1] Aarts and Fokkink showed that if $f$ is a $k$-to-1 self-map of the 2-adic solenoid then $k$ is odd. It would be interesting to know the answer to the following.

**Question 4.** If $S$ is a $P$-adic pseudosolenoid, when is there a $k$-to-1 self-map of $S$? Must $k$ be coprime with all but finitely many $p_i$? What if $S$ is a $P$-adic solenoid?

**Question 5.** If $S$ is a $P$-adic pseudosolenoid, is every exactly $k$-to-1 map on $S$ a $k$-fold covering map?

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References


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